Quantum Gate Fidelity in Terms of Choi Matrices

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Joint work with David W. Kribs

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Quantum Information Theory Background

- Pure quantum states are represented by unit vectors $|\phi\rangle \in \mathbb{C}^n$.
- Ideally, quantum gates are represented by unitary transformations on $\mathbb{C}^n$.
- In practice, however, things aren't always unitary – noise gets introduced into the system.
- In practice, quantum channels are represented by completely positive, trace-preserving maps.

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- A **quantum channel** $\mathcal{Q}$ is a completely positive, trace-preserving map on M_n. That is,
  1. $\mathcal{Q}(X) \in M_n^+ \quad \forall X \in M_n^+$ (i.e., $\mathcal{Q}$ is **positive**);
  2. $id_m \otimes \mathcal{Q}$ is positive for all $m \geq 1$; and
  3. $\mathrm{Tr}(\mathcal{Q}(X)) = \mathrm{Tr}(X) \quad \forall X \in M_n$.

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- To determine whether or not a map Λ is completely positive, it is enough to determine whether or not its **Choi matrix** is positive semidefinite.
- The Choi matrix of a map Λ on M_n is the following operator on $\mathbb{C}^n \otimes \mathbb{C}^n$:

$$C_\Lambda := \sum_{i,j=1}^{n} |i\rangle\langle j| \otimes \Lambda\big(|i\rangle\langle j|\big).$$

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What is Quantum Gate Fidelity?

Quantum gate fidelity is a way of measuring the distance between the unitary channel $\mathcal{U}$ we *wanted* to implement and the quantum channel $\mathcal{Q}$ that we *did* implement.

- It is a function on pure states (unit vectors in $\mathbb{C}^n$) defined by:

$$\begin{aligned}\mathcal{F}_{\mathcal{Q},\mathcal{U}}(|\phi\rangle) &:= \mathrm{Tr}\big(\mathcal{Q}(|\phi\rangle\langle\phi|)\mathcal{U}(|\phi\rangle\langle\phi|)\big) \\ &= \mathrm{Tr}\big(\mathcal{U}^\dagger \circ \mathcal{Q}(|\phi\rangle\langle\phi|)|\phi\rangle\langle\phi|\big)\end{aligned}$$

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Some Simple Examples

Recall that

$$\mathcal{F}_{\mathcal{Q}}(|\phi\rangle) = \mathrm{Tr}\big(\mathcal{Q}(|\phi\rangle\langle\phi|)|\phi\rangle\langle\phi|\big).$$

- It is clear that $\mathcal{F}_{\mathcal{Q}} \equiv \mathcal{F}_{\mathcal{Q}^\dagger}$, where $\mathcal{Q}^\dagger$ is the dual map defined by

$$\mathrm{Tr}\big(\mathcal{Q}(X)Y\big) = \mathrm{Tr}\big(X\mathcal{Q}^\dagger(Y)\big) \quad \forall X, Y \in M_n.$$

- A bit more generally, if $\mathcal{E}$ is a unital channel (i.e., $\mathcal{E}(I) = I$) and $r \geq 0$ then $\mathcal{Q} + r(\mathcal{E} - \mathcal{E}^\dagger)$ has the same gate fidelity as $\mathcal{Q}$ itself.

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What about the converse?

If $\mathcal{F}_{\mathcal{Q}} \equiv \mathcal{F}_{\mathcal{R}}$, does there exist $r \geq 0$ and a unital channel $\mathcal{E}$ such that $\mathcal{R} = \mathcal{Q} + r(\mathcal{E} - \mathcal{E}^\dagger)$?

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In 4 or More Dimensions

It was shown in [Magesan, 2010] that the converse does not hold when $n \geq 4$.

To see this, a specific map Λ on M_4 was constructed with three properties:

1. $\mathcal{F}_\Lambda \equiv 0$;
2. Λ is trace-preserving; and
3. There does not exist $r \geq 0$ and a unital channel $\mathcal{E}$ with $\Lambda = r(\mathcal{E} - \mathcal{E}^\dagger)$.

Then $\mathcal{F}_\mathcal{Q} \equiv \mathcal{F}_{\mathcal{Q}+\varepsilon\Lambda}$.

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In 2 or 3 Dimensions

In contrast with the case when the dimension is 4 or higher, we have the following result in 2 or 3 dimensions:

Theorem

Let $\mathcal{Q}$ and $\mathcal{R}$ be quantum channels on M_n. Suppose that either

- $n = 2$; *or*
- $n = 3$ *and* $\mathcal{Q}(I) = \mathcal{R}(I)$.

Then $\mathcal{F}_{\mathcal{Q}} \equiv \mathcal{F}_{\mathcal{R}}$ if and only if there exists $r \geq 0$ and a unital quantum channel $\mathcal{E}$ such that $\mathcal{R} = \mathcal{Q} + r(\mathcal{E} - \mathcal{E}^\dagger)$.

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General Characterization

We now present a characterization of when channels have identical gate fidelity in arbitrary dimension. But first...

- The **symmetric subspace** $\mathcal{S} \subset \mathbb{C}^n \otimes \mathbb{C}^n$ is the subspace spanned by states of the form $|i\rangle \otimes |j\rangle + |j\rangle \otimes |i\rangle$.
- We denote the projection onto the symmetric subspace by $P_{\mathcal{S}}$.
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General Characterization

As a special case of the characterization of gate fidelity, consider the case of quantum channels with constant gate fidelity.

- It is well-known that depolarizing channels (i.e., channels of the form $\mathcal{Q}(X) \equiv pX + \frac{(1-p)\mathrm{Tr}(X)}{n} I$) have constant gate fidelity.
- There are others that have constant gate fidelity when $n \geq 4$ as well...

Corollary

Let $\mathcal{Q}$ be a quantum channel and let $c \in \mathbb{R}$. Then $\mathcal{F}_{\mathcal{Q}} \equiv c$ if and only if $P_{\mathcal{S}} C_{\mathcal{Q}}^{\Gamma} P_{\mathcal{S}} = cP_{\mathcal{S}}$.

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Measures Based on Gate Fidelity

The gate fidelity function is often modified in some way to provide a single number that can be used as a measure the gate fidelity of a channel $\mathcal{Q}$.

- One way of obtaining a single number from the gate fidelity is to average its value over all pure states.
- Alternatively, one may minimize the gate fidelity function over all pure states.
- Because all of the information about the gate fidelity of $\mathcal{Q}$ is contained in $P_S C_{\mathcal{Q}}^{\Gamma} P_S$, it is instructive to see how the average and minimum gate fidelity depend on this operator.

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Average Gate Fidelity

The **average gate fidelity** $\overline{\mathcal{F}_{\mathcal{Q}}}$ of a quantum channel $\mathcal{Q}$ is the average of $\mathcal{F}_{\mathcal{Q}}(|\phi\rangle)$ over all pure states $|\phi\rangle$.

The average gate fidelity is known to be simple to calculate in terms of the Kraus operators of $\mathcal{Q}$. The following proposition is a simple rephrasing of a well-known formula.

Proposition

Let $\mathcal{Q}$ be a quantum channel on M_n. Then

$$\overline{\mathcal{F}_{\mathcal{Q}}} = \frac{n + \mathrm{Tr}\big(P_{\mathcal{S}} C_{\mathcal{Q}}^{\Gamma} P_{\mathcal{S}}\big)}{n(n+1)}.$$

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Minimum Gate Fidelity

The **minimum gate fidelity** $\mathcal{F}_{\mathcal{Q}}^{min}$ of a quantum channel $\mathcal{Q}$ is the minimum of $\mathcal{F}_{\mathcal{Q}}(|\phi\rangle)$ over all pure states $|\phi\rangle$.

The minimum gate fidelity is expected to be tough to calculate. In order to present it in terms of $P_{\mathcal{S}}C_{\mathcal{Q}}^{\Gamma}P_{\mathcal{S}}$, we first define a norm on $M_n \otimes M_n$:

Let $X \in M_n \otimes M_n$. Then we define

$$\big\|X\big\|_{S(1)} := \sup_{|v\rangle,|w\rangle} \Big\{ \big|\langle v|X|w\rangle\big| : |v\rangle, |w\rangle \text{ are separable} \Big\}.$$

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Theorem

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$$\mathcal{F}_{\mathcal{Q}}^{min} = \lambda - \left\| \lambda P_{\mathcal{S}} - P_{\mathcal{S}} C_{\mathcal{Q}}^{\Gamma} P_{\mathcal{S}} \right\|_{S(1)}.$$

- It is NP-HARD to compute $\| \cdot \|_{S(1)}$ on general positive operators. This doesn't imply that $\mathcal{F}_{\mathcal{Q}}^{min}$ is NP-HARD, but it certainly seems suggestive.
- Nevertheless, much is known about $\| \cdot \|_{S(1)}$ that we can now immediately apply to minimum gate fidelity.

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For example, we immediately get the following bounds:

Corollary

Let $\mathcal{Q}$ be a quantum channel on M_n. Denote the eigenvalues of $P_S C_{\mathcal{Q}}^{\Gamma} P_S$ supported on P_S by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n(n+1)/2}$ (i.e., these are the $n(n+1)/2$ potentially nonzero eigenvalues of $P_S C_{\mathcal{Q}}^{\Gamma} P_S$). Let α_j be the maximal Schmidt coefficient of the eigenvector corresponding to λ_j. Then

$$\mathcal{F}_{\mathcal{E}}^{min} \leq \lambda_1 - \max_j \{(\lambda_1 - \lambda_j)\alpha_j^2\} \quad \textit{and}$$

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Minimum Gate Fidelity

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Minimum Gate Fidelity

Also, $\|\cdot\|_{S(1)}$ can be computed when $n = 2$ via semidefinite programming, which allows us to compute $\mathcal{F}_{\mathcal{Q}}^{min}$ for qubit channels.

The semidefinite program is as follows, where we optimize over states ρ:

$$\begin{aligned} \text{minimize:} \quad & \lambda_1 - \mathrm{Tr}\big((\lambda_1 P_{\mathcal{S}} - P_{\mathcal{S}} C_{\mathcal{Q}}^{\Gamma} P_{\mathcal{S}})\rho\big) \\ \text{subject to:} \quad & \rho \in (M_2 \otimes M_2)^+ \\ & \rho^{\Gamma} \in (M_2 \otimes M_2)^+ \\ & \mathrm{Tr}(\rho) = 1 \end{aligned}$$

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Further Reading

- E. Magesan, *Depolarizing behavior of quantum channels in higher dimensions*. arXiv:1002.3455 [quant-ph]
- N. J., D. W. Kribs, *Quantum gate fidelity in terms of Choi matrices*. arXiv:1102.0948 [quant-ph]