Minimal and Maximal Operator Spaces and Operator Systems in Entanglement Theory

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Order of Events

- Introduction to (Minimal and Maximal) Operator Spaces
- Connection to Entanglement Theory
- Introduction to (Minimal and Maximal) Operator Systems
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What is an Operator Space?

An (abstract) operator space is a normed vector space V together with a family of norms $\|\cdot\|_{M_m(V)}$ on $M_m(V) \cong M_m \otimes V$ (one norm for each $m \in \mathbb{N}$) that satisfy two properties, both of which are inspired by the standard operator norm on matrices:

1. If $X \in M_m(V)$ and $Y \in M_p(V)$ then

$$\left\|X \oplus Y\right\|_{M_{m+p}(V)} = \max\left\{\|X\|_{M_m(V)}, \|Y\|_{M_p(V)}\right\}.$$

This is called the L^∞ **property**.

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2. The second property is analogous to the submultiplicativity of the operator norm:

   If $A = (a_{ij}), B = (b_{ij}) \in M_{p,m}$ and $X = (x_{ij}) \in M_m(V)$ then $\|A \cdot X \cdot B^*\|_{M_p(V)} \leq \|A\|\|X\|_{M_m(V)}\|B\|$, where

$$(A \cdot X)_{ij} := \sum_{k=1}^{m} a_{ik} x_{kj}.$$

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We will be particularly interested in the case when the normed vector space V that we're dealing with is M_n, the $n \times n$ complex matrices with the operator norm.

- In this case, it is easy to construct a family of norms on $M_m(M_n)$ just by making the identification $M_m(M_n) \cong M_{mn}$ and using the operator norm.
- This "naive" family of norms satisfies the L^∞ and submultiplicative properties. What are some more "exotic" operator space norms on M_n? Can we always construct an operator space out of any normed vector space V?

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The Minimal Operator Space

For $X = (x_{ij}) \in M_m(V)$ consider the following norms, called the **minimal operator space norms**:

$$\left\|X\right\|_{M_m(MIN(V))} := \sup\left\{\left\|(f(x_{ij}))\right\| : f : V \to \mathbb{C} \text{ with } \|f\| \leq 1\right\}.$$

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The Minimal Operator Space

- The minimal operator space norms are the smallest operator space norms on V that exist.
- That is, if $\|\cdot\|_{M_m(V)}$ is another family of operator space norms on V then

$$\big\|X\big\|_{M_m(MIN(V))} \leq \big\|X\big\|_{M_m(V)} \quad \forall\, X \in M_m(V).$$

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where the supremum is taken over all Hilbert spaces $\mathcal{H}$ and all maps Φ.

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k-Minimal and k-Maximal Operator Spaces

What if, instead of starting just with a norm on V, we started with norms on $V, M_2(V), M_3(V), \ldots,$ and $M_k(V)$ that satisfy the L^∞ and submultiplicativity properties?

- Is it always possible to "complete" this family of norms and turn V into an operator space?
- In other words, is it always possible to define norms on $M_{k+1}(V), M_{k+2}(V), \ldots$ such that V becomes an operator space?
- Yes, and the method of doing it is analogous to the construction of the minimal and maximal operator spaces we already saw.

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- If $m \leq k$ then

$$\big\|X\big\|_{M_m(MIN^k(V))} = \big\|X\big\|_{M_m(V)} = \big\|X\big\|_{M_m(MAX^k(V))}.$$

- Given any other operator space norms $\|\cdot\|_{M_m(O(V))}$ on V, if $\|\cdot\|_{M_m(O(V))} = \|\cdot\|_{M_m(V)}$ for $1 \leq m \leq k$ then

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Separable States and Schmidt Rank

In quantum information theory, a bipartite pure state is typically represented by a unit vector in $\mathbb{C}^m \otimes \mathbb{C}^n$.

- We will use "kets" $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ to represent unit vectors (pure states).
- $|v\rangle$ is called **separable** if there exist $|x\rangle \in \mathbb{C}^m, |y\rangle \in \mathbb{C}^n$ such that $|v\rangle = |x\rangle \otimes |y\rangle$.
- We say that the **Schmidt rank** of $|v\rangle$ (written $SR(|v\rangle)$) equals k if we can write $|v\rangle$ as a linear combination of k (but not fewer) separable pure states.
- $1 \leq SR(|v\rangle) \leq \min\{m, n\}$ always, and $SR(|v\rangle) = 1$ if and only if $|v\rangle$ is separable.

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S(k)-Norms

Consider the following family of norms for $X \in M_m \otimes M_n$ based on Schmidt rank:

$$\big\|X\big\|_{S(k)} := \sup\Big\{\big|\langle v|X|w\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k\Big\}.$$

- These norms are useful for trying to detect k-block-positivity of operators (and hence k-positivity of linear maps).
- Computing these norms on a particular family of projections would resolve the NPPT bound entanglement conjecture.
- A conjecture about the $S(1)$-norm would imply that the regularized relative entropy of entanglement is super-additive and that $QMA(k) = QMA(2)$ for all $k > 2$.

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Connection to k-Minimal Operator Spaces

Theorem

Let $X \in M_m(M_n)$. *Then*

$$\big\|X\big\|_{S(k)} = \big\|X\big\|_{M_m(MIN^k(M_n))}.$$

What is an Operator System?

An (abstract) operator system is an ordered $*$-vector space V together with a family of cones C_m in $M_m(V)$ (one cone for each $m \in \mathbb{N}$) that satisfy three properties, all of which are inspired by the cone of positive semidefinite matrices:

1. Each C_m is a *proper* cone (i.e., $C_m \cap -C_m = \{0\}$) in $M_m(V)_h$.
2. There exists $\mathbb{I}_m \in C_m$ such that for any $v \in M_m(V)_h$, there exists $r > 0$ such that $r\mathbb{I}_m - v \in C_m$.

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3. If $A = (a_{ij}) \in M_{p,m}$ and $X = (x_{ij}) \in C_m$ then $A \cdot X \cdot A^* \in C_p$, where

$$(A \cdot X)_{ij} := \sum_{k=1}^{m} a_{ik} x_{kj}.$$

4. Technically, we also require that each $M_m(V)$ be Archimedean, but this is a technical requirement needed only in infinite dimensions (which we will avoid).

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We will be particularly interested in the case when the ordered $*$-vector space V that we're dealing with is M_n, the $n \times n$ complex matrices with the usual cone of positive semidefinite matrices C_1.

- In this case, it is easy to construct a family of cones in $M_m(M_n)$ just by making the identification $M_m(M_n) \cong M_{mn}$ and choosing the cone of positive semidefinite matrices.
- This "naive" family of cones satisfies the operator system axioms. What are some more "exotic" operator system cones on M_n? Can we always construct an operator system out of any Archimedean ordered $*$-vector space V?

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k-Super Minimal and k-Super Maximal Operator Spaces

What if, instead of starting just with a cone of positive elements $C_1 \subset V$, we started with cones $C_1 \subset V$, $C_2 \subset M_2(V)$, $C_3 \subset M_3(V), \ldots,$ and $C_k \subset M_k(V)$ that satisfy the operator system axioms?

- Is it always possible to "complete" this family of cones and turn V into an operator system?
- In other words, is it always possible to define cones in $M_{k+1}(V), M_{k+2}(V), \ldots$ such that V becomes an operator system?
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Separable Mixed States and Schmidt Number

In quantum information theory, not all states are pure – some are *mixed*. A general (mixed, bipartite) quantum state is represented as a **density matrix** $\rho \in M_m \otimes M_n$.

- Density matrices have trace 1 and are positive semidefinite.
- Any density matrix ρ can be written as

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where each $|v_i\rangle$ is a pure state and $\{p_i\}$ is a probability distribution.

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- The **Schmidt number** of a (bipartite, mixed) state ρ (written $SN(\rho)$) equals k if we can write ρ in the form

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with $SR(|v_i\rangle) \leq k$ for each i, but we can't write it in that form with $SR(|v_i\rangle) < k$ for each i.

- $1 \leq SN(\rho) \leq \min\{m, n\}$ always.
- If $SN(\rho) = 1$ then ρ is called **separable**.

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A matrix $X \in M_m \otimes M_n$ is called k-**block positive** if

$$\langle v|X|v\rangle \geq 0 \quad \forall |v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^m \text{ with } SR(|v\rangle) \leq k.$$

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- k-block positive operators are dual to the cone of (unnormalized) states with Schmidt number at most k.
- That is, X is k-block positive if and only if

$$\mathrm{Tr}(X\rho) \geq 0 \quad \forall\, \rho \in M_m \otimes M_n \text{ with } SN(\rho) \leq k.$$

- Similarly, $SN(\rho) \leq k$ if and only if

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- This final condition leads to k-block positive operators sometimes being called **k-entanglement witnesses**.

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Connection to Minimal and Maximal Operator Systems

Theorem

Let $X, \rho \in M_m \otimes M_n$. Then

(a) $X \in C_m^{min,k}$ *if and only if* X *is k-block positive; and*

(b) $\rho \in C_m^{max,k}$ *if and only if* $SN(\rho) \leq k$.

Further Reading

- V. I. Paulsen, *Completely bounded maps and operator algebras*. Cambridge University Press, Cambridge (2003).
- T. Oikhburg, E. Ricard, *Operator spaces with few completely bounded maps*. Math. Ann. **328**, 229-259 (2004).
- V. I. Paulsen, I. Todorov, M. Tomforde, *Operator system structures on ordered spaces*. Proc. of the LMS (to appear). arXiv:0904.3783
- N. Johnston, D. W. Kribs, V. I. Paulsen, and R. Pereira, *Minimal and Maximal Operator Spaces and Operator Systems in Entanglement Theory*. To appear in Journal of Functional Analysis (2010). arXiv:1010.1432