Minimal and Maximal Operator Spaces and Operator Systems in Entanglement Theory

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Can operator spaces and operator systems help us better-understand quantum entanglement?

- Complete positivity between operator systems leads to the well-known positivity-based separability criteria.
- Complete boundedness between operator spaces leads to the more recently-developed norm-based separability criteria.

Operator Systems

An **operator system** on M_n is a family of "positive" cones $C_p \subseteq M_p(M_n) \cong M_p \otimes M_n$ that agree with M_n^+, the cone of positive semidefinite matrices, in the following two ways:

- $C_1 = M_n^+$, and
- $(A \otimes I)^* C_p (A \otimes I) \subseteq C_q$ for all $A \in M_{p,q}$.

The most well-known operator system is given by the "naive" family of cones obtained by associating $M_p(M_n)$ with M_{pn} and defining $C_p := M_{pn}^+$.

OMIN and OMAX

Among all operator systems, there is a **minimal** and a **maximal** one [1], denoted $OMIN(M_n)$ and $OMAX(M_n)$, respectively.

- The cones defining $OMIN(M_n)$ are the entanglement witnesses in $M_p \otimes M_n$.
- The cones defining $OMAX(M_n)$ are the (unnormalized) separable states.

Complete Positivity

A map $\Phi : O_1(M_n) \rightarrow O_2(M_n)$ between operator systems, defined by cones $\{C_p\}$ and $\{D_p\}$ respectively, is called **completely positive** if $(id_p \otimes \Phi)(C_p) \subseteq D_p$ for all p.

- If $O_1(M_n) = O_2(M_n) = M_n$, the "naive" operator system, then this is the standard notion of complete positivity that quantum information theorists know and love.
- $\Phi : OMAX(M_n) \rightarrow M_n$ is completely positive if and only if Φ is positive.
- $\Phi : OMIN(M_n) \rightarrow M_n$ is completely positive if and only if Φ is entanglement-breaking.

Separability criteria in terms of positive maps can be rephrased in terms of minimal and maximal operator systems.

Separability as Complete Positivity

$$\rho \text{ separable } \Leftrightarrow (id_n \otimes \Phi)(\rho) \geq 0 \quad \forall \Phi \text{ positive}$$
$$\updownarrow$$
$$\Phi \in CP(OMAX(M_n), M_n) \Leftrightarrow \Phi \text{ is positive}$$
$$\updownarrow$$
$$\Phi \in CP(OMIN(M_n), M_n) \Leftrightarrow \Phi \text{ is e.b.}$$

Operator Spaces

An **operator space** on M_n is a family of norms $\|\cdot\|_p$ on $M_p(M_n) \cong M_p \otimes M_n$ that agree with the operator norm in the following two ways:

- $\|X \oplus Y\|_{p+q} = \max\{\|X\|_p, \|Y\|_q\}$ for all $X \in M_p(M_n), Y \in M_q(M_n)$, and
- $\|(A \otimes I)^* X (B \otimes I)\|_q \leq \|A\|\|X\|_p\|B\|$ for all $A, B \in M_{p,q}$, $X \in M_p \otimes M_n$.

The most well-known operator space is given by the "naive" family of norms obtained by associating $M_p(M_n)$ with M_{pn} and defining $\|\cdot\|_p$ to be the operator norm on M_{pn}.

MIN and MAX

Among all operator spaces, there is a **minimal** and a **maximal** one [2, Chapter 14], denoted $MIN(M_n)$ and $MAX(M_n)$, respectively.

- The norms on $MIN(M_n)$ are given by $\|X\|_p = \sup\{|\langle v|X|w\rangle|\}$, where the supremum is taken over separable pure states $|v\rangle$ and $|w\rangle$.
- The MAX operator system lets us get our hands on the duals of the $MIN(M_n)$ norms, which are given by
$$\|X\|_p' = \inf\{\mathrm{Tr}(\sqrt{\Sigma_i A_i A_i^*})\mathrm{Tr}(\sqrt{\Sigma_i B_i B_i^*})\},$$
where the infimum is taken over all decompositions $X = \Sigma_i A_i B_i^* \otimes X_i$ with $\|X_i\|_{tr} \leq 1$.

Complete Boundedness

The **completely bounded norm** of a map $\Phi : O_1(M_n) \rightarrow O_2(M_n)$ between operator spaces, defined by norms $\|\cdot\|_p$ and $\|\cdot\|_p'$ respectively, is computed as $\sup_{p,X}\{\|(id_p \otimes \Phi)(X)\|_p'/\|X\|_p\}$.

- If $O_1(M_n) = O_2(M_n) = M_n$, the "naive" operator space, then this is the standard completely bounded norm that is the dual of the diamond norm.
- By relating the $OMIN$ to MIN and $OMAX$ to MAX, we are able to use completely bounded norms to characterize separability.

Separability can be characterized via maps Φ that are trace-contractive on Hermitian operators ($\|\Phi\|_{tr}^H \leq 1$) [3].

- This approach to separability testing can be seen as arising from completely bounded norms.
- The CB-norm approach generalizes easily to Schmidt number larger than 1 via k-**minimal** and k-**maximal** operator spaces [4].

Separability as Complete Boundedness

$$\rho \text{ separable } \Leftrightarrow \|(id_n \otimes \Phi)(\rho)\|_{tr} \leq 1 \quad \forall \|\Phi\|_{tr}^H \leq 1$$
$$\updownarrow$$
$$\|\Phi\| = \|\Phi\|_{CB(M_n, MIN(M_n))}$$

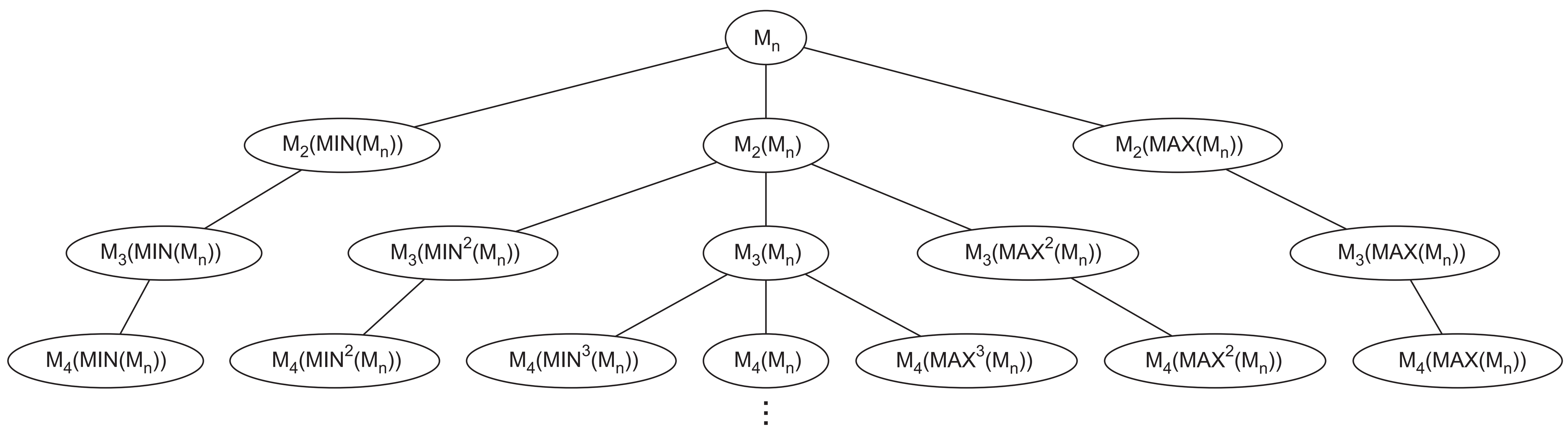

Fig. 1: Each simple path starting at the root node M_n represents one of the k-minimal or k-maximal operator spaces. The leftmost path gives $MIN(M_n)$ and the rightmost path gives $MAX(M_n)$. The path down the centre gives the "naive" operator space M_n itself.

Conclusions

Using operator spaces and operator systems we have

- found a link between positivity-based and norm-based separability criteria, and
- shown that some recently-discovered ideas in entanglement theory are simple when viewed from operator theory.

For Further Information

For the details of our work:

- N. Johnston, D. W. Kribs, V. I. Paulsen, and R. Pereira, *Minimal and maximal operator spaces and operator systems in entanglement theory*. Journal of Functional Analysis (to appear, 2011). arXiv:1010.1432

Preprints and this poster can be downloaded from:

- www.arxiv.org/abs/1010.1432
- www.nathanieljohnston.com

References

[1] V. I. Paulsen, I. Todorov, M. Tomforde, *Operator system structures on ordered spaces*. Proc. of the LMS (to appear).

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