Applications of a Family of Norms in Entanglement Theory

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Outline

Several problems in quantum information theory can be phrased in terms of a family of operator norms that we will discuss today...

- Determining whether or not an operator is an entanglement witness (or a k-entanglement witness).
- The NPPT bound entanglement problem.
- The minimum gate fidelity of a quantum channel.

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Notation

- $\mathcal{H}$ is an (n-dimensional) complex Hilbert space. The space of linear operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$ (or just $\mathcal{L}$ for short).
- The Schmidt rank (a.k.a. tensor rank) of the bipartite pure state $|v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ will be written $SR(|v\rangle)$.
- If $SR(|v\rangle) = 1$ (i.e., $|v\rangle = |a\rangle \otimes |b\rangle$) then $|v\rangle$ is called separable.
- Recall that $1 \leq SR(|v\rangle) \leq n$.

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- Not all quantum states are pure – we will sometimes consider mixed states, which are represented by density operators.
- A density operator $\rho \in \mathcal{L}$ is a positive semidefinite operator such that $\mathrm{Tr}(\rho) = 1$.
- By the spectral decomposition, we can always write mixed states as a convex combination of projections onto pure states:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i|.$$

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$S(k)$-Norms

Let $X \in \mathcal{L}_A \otimes \mathcal{L}_B$ and let $1 \leq k \leq n$. Then we define the $S(k)$**-norm** of X by

$$\big\|X\big\|_{S(k)} := \sup_{|v\rangle,|w\rangle} \Big\{ \big|\langle w|X|v\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k \Big\}.$$

- Yes, these are actually norms.
- If $k = n$, then this is the standard operator norm. That is, $\big\|X\big\|_{S(n)} = \big\|X\big\|$.
- $\big\|X\big\|_{S(1)} \leq \big\|X\big\|_{S(2)} \leq \cdots \leq \big\|X\big\|_{S(n-1)} \leq \big\|X\big\|$

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Some more facts about the $S(k)$-norms...

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- If X is positive semidefinite (the case we will be most interested in), we can always choose $|w\rangle = |v\rangle$ – not true in general for normal (or even Hermitian) operators though.

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(Un)distillable States

Suppose Alice and Bob share a state $\rho_{AB} \in \mathcal{L}_A \otimes \mathcal{L}_B$, and they want to extract a maximally entangled pure state from it. However, they are only able to perform local quantum operations and classical communication (LOCC).

- That is, they want to use an LOCC operation Φ so that $\Phi(\rho) = |e\rangle\langle e|$, where $|e\rangle = \frac{1}{\sqrt{n}} \sum_i |i\rangle_A \otimes |i\rangle_B$.
- If such an LOCC operation exists, ρ_{AB} is called **distillable**.
- Otherwise, ρ_{AB} is called **undistillable**.

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- If such an LOCC operation exists, ρ_{AB} is called ***r*-distillable**.
- Otherwise, ρ_{AB} is called ***r*-undistillable**.
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Some facts about (un)distillable states:

- If ρ_{AB} is r-distillable, then it is $(r+1)$-distillable.
- If ρ_{AB} is separable, then it is bound entangled.
- If ρ_{AB} has positive partial transpose (i.e., $\rho_{AB}^\Gamma \geq 0$), then it is bound entangled.

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It is now over 10 years later and we still don't know.

Removing Mention of LOCC

Distillability and bound entanglement can be phrased (relatively) simply in terms of the partial transpose and vectors with Schmidt rank 2, removing the ugly need to discuss LOCC operations.

- ρ_{AB} is undistillable if and only if

$$\langle v|\rho_{AB}^{\Gamma}|v\rangle \geq 0 \quad \forall |v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \text{ with } SR(|v\rangle) \leq 2.$$

- If $\rho_{AB}^{\Gamma} \not\geq 0$ (the case we are interested in), then this is equivalent to saying that ρ_{AB}^{Γ} is a **2-entanglement witness** – it is positive on states with Schmidt rank no larger than 2, but it is not positive on all states.

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Do there exist density operators ρ_{AB} such that ρ_{AB}^Γ is a 2-entanglement witness?

- ρ_{AB} is NPPT r-undistillable if and only if $(\rho_{AB}^{\otimes r})^\Gamma$ is a 2-entanglement witness.
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Werner States

For the NPPT bound entanglement problem, it has been shown that it is enough to consider only **Werner states**.

- Let $S \in \mathcal{L}_A \otimes \mathcal{L}_B$ be the swap operator that maps $|a\rangle \otimes |b\rangle$ to $|b\rangle \otimes |a\rangle$. Werner states are the density operators of the following form:

$$\rho_\alpha := I - \alpha S \in \mathcal{L}_A \otimes \mathcal{L}_B \quad \text{for some } \alpha \in [-1, 1].$$

- Really, we should have a scaling factor of $\frac{1}{n^2 - \alpha n}$ in front of the Werner state so that $\mathrm{Tr}(\rho_\alpha) = 1$, but we will ignore it as it does not affect bound entanglement.

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It is not difficult to verify that ρ_α^Γ is a 2-entanglement witness if and only if $\frac{1}{n} < \alpha \leq \frac{1}{2}$.

- The NPPT bound entanglement conjecture is then equivalent to asking whether or not there exists $\frac{1}{n} < \alpha \leq \frac{1}{2}$ such that $(\rho_\alpha^\Gamma)^{\otimes r}$ is a 2-entanglement witness for all $r \geq 1$.
- Numerical evidence suggests that ρ_α is indeed bound entangled for all $\frac{1}{n} < \alpha \leq \frac{1}{2}$.

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Relationship with the $S(2)$-Norm

Recall that for $X \in (\mathcal{L}_A \otimes \mathcal{L}_B)^+$, we have

$$\big\|X\big\|_{S(2)} := \sup_{|v\rangle} \Big\{ \langle v|X|v\rangle : SR(|v\rangle) \leq 2 \Big\}.$$

Proposition

Suppose $X = X^* \in \mathcal{L}_A \otimes \mathcal{L}_B$ has exactly one positive eigenvalue λ and exactly one negative eigenvalue μ, and let $P \in \mathcal{L}_A \otimes \mathcal{L}_B$ denote the projection onto the negative eigenspace of X. Then X is 2-entanglement witness if and only if $\|P\|_{S(2)} \leq \frac{\lambda}{\lambda-\mu}$.

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Relationship with the $S(2)$-Norm

Let's apply the proposition to Werner states! The partial transpose of a Werner state has the form

$$\rho_\alpha^\Gamma = I - \alpha n|e\rangle\langle e|,$$

where $|e\rangle = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}|i\rangle_A \otimes |i\rangle_B$ is the "standard" pure maximally entangled state.

- When $\alpha = \frac{2}{n}$, the eigenvalues of ρ_α^Γ are simply 1 and -1.
- Then for any $r \geq 1$, the eigenvalues of $(\rho_{2/n}^{\otimes r})^\Gamma$ are also 1 and -1.

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It follows that bound entanglement of $\rho_{2/n}$ can be determined by examining the $S(2)$-norm of the projections onto the negative eigenspace of $(\rho_{2/n}^{\otimes r})^\Gamma$.

Theorem

For $n \geq 4$, the state $\rho_{2/n}$ is r-undistillable if and only if $\big\|P^{(r)}\big\|_{S(2)} \leq \frac{1}{2}$, where $P^{(r)}$ is the orthogonal projection defined recursively via

$$P^{(1)} := |e\rangle\langle e|_{AB},$$

$$P^{(r+1)} := P_{AB}^{(1)} \otimes (I - P^{(r)})_{A'B'} + (I - P^{(1)})_{AB} \otimes P_{A'B'}^{(r)}, \text{ for } r \geq 1.$$

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Relationship with the $S(2)$-Norm

Computing the $S(2)$-norm on these projections is tricky. What **do** we know so far?

- $\|P^{(r)}\|_{S(1)} = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{2}{n}\right)^r$
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As a corollary of the fact that $\left\| P^{(r)} \right\|_{S(2)} \leq 1 - \left(1 - \frac{2}{n}\right)^r$, we obtain the following partial result:

Corollary

If $\frac{1}{n} < \alpha \leq \min\left\{\frac{2}{n}, \frac{\ln(2)}{r+3\ln(2)-1}\right\}$ then ρ_α is NPPT r-undistillable.

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Minimum Gate Fidelity – Definition

The **gate fidelity** of a quantum channel (completely positive trace-preserving map) $\mathcal{E} : \mathcal{L} \to \mathcal{L}$ and a unitary channel $\mathcal{U}(\rho) = U\rho U^\dagger$ is a function on pure states defined by

$$\mathcal{F}_{\mathcal{E},\mathcal{U}}(|v\rangle) = \mathrm{Tr}\big(\mathcal{E}(|v\rangle\langle v|)\mathcal{U}(|v\rangle\langle v|)\big).$$

- Without loss of generality, we can assume $U = I$ and we will simply write $\mathcal{F}_{\mathcal{E}}(|v\rangle)$.
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Minimum Gate Fidelity – Properties

Minimum gate fidelity satisfies many nice properties that make it a good tool for measuring how close $\mathcal{E}$ is to the unitary channel $\mathcal{U}$.

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Choi Matrices

Before proceeding, we will need to briefly introduce the Choi matrix of a quantum channel.

- The **Choi matrix** of a quantum channel $\mathcal{E}$ is the operator

$$C_{\mathcal{E}} := n(id_n \otimes \mathcal{E})(|e\rangle\langle e|).$$

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The Symmetric Subspace

We will also need to consider the **symmetric subspace**:

- The symmetric subspace $\mathcal{S} \subseteq \mathcal{H}_A \otimes \mathcal{H}_B$ is the span of vectors of the form $|v\rangle \otimes |v\rangle$.
- Equivalently, it is the set of vectors that are fixed under the action of the swap operator S.
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Relationship with the $S(1)$-Norm

We are now ready to connect the minimum gate fidelity to the $S(1)$-norm:

Theorem

Define $\lambda := \|P_{\mathcal{S}} C_{\mathcal{E}}^{\Gamma} P_{\mathcal{S}}\|$. Then

$$\mathcal{F}_{\mathcal{E}}^{min} = \lambda - \left\| P_{\mathcal{S}}(\lambda I - C_{\mathcal{E}}^{\Gamma}) P_{\mathcal{S}} \right\|_{S(1)}.$$

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Relationship with the $S(1)$-Norm

Now we can apply everything that we know about the $S(1)$-norm to minimum gate fidelity.

- In the case when $n = 2$, we can quickly compute the $S(1)$-norm via semidefinite programming. Thus we can now compute $\mathcal{F}_{\mathcal{E}}^{min}$ for qubit channels.
- In general, we can upper bound the $S(1)$-norm via semidefinite programming as well, which allows us to get lower bounds for $\mathcal{F}_{\mathcal{E}}^{min}$.

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Further Reading

N. J., D. W. Kribs, *A Family of Norms With Applications In Quantum Information Theory*. arXiv:0909.3907 [quant-ph]

N. J., D. W. Kribs, *A Family of Norms With Applications In Quantum Information Theory II*. arXiv:1006.0898 [quant-ph]

N. J., D. W. Kribs, *Quantum Gate Fidelity in Terms of Choi Matrices*. arXiv:1102.0948 [quant-ph]