SCHMIDT NORMS FOR QUANTUM STATES

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Abstract. We consider a family of vector and operator norms which we refer to as Schmidt norms. We show that these norms have several uses in quantum information theory – they can be used to help classify k-positive linear maps (and hence entanglement witnesses), they are useful for approaching the problem of finding non-positive partial transpose bound entangled Werner states, they are related to the quantum fidelity and trace distance measures, and they are connected to the recently-defined local numerical range. We show that the vector norms can be explicitly calculated, and we derive several inequalities in order to bound the operator norms and compute them in special cases. We show that one particular entangled Werner state is bound entangled if and only if a certain norm inequality holds on a given family of projections, and we use our inequalities to study that family of projections. We also develop a family of semidefinite programs that can be used to further bound the operator norms. We extend these norms to arbitrary convex mapping cones and explore their implications with positive partial transpose states.

1. Introduction

The need to develop local analogues of tools from matrix analysis and linear algebra for states on bipartite (and multipartite) quantum systems is of paramount importance in quantum information theory. In particular, one is often interested in properties of an operator or state when restricted to one system or another, or how an operator can change under local operations on one of the component Hilbert spaces. In this paper, we conduct a detailed analysis of a family of vector and operator norms that we refer to as *Schmidt norms*, since they derive from the fundamental Schmidt decomposition theorem for quantum states [1]. We also present a number of applications of these norms in quantum information theory.

The Schmidt norms generalize the standard Euclidean and operator norms and can be regarded as the local analogues of these norms. More specifically, the Schmidt vector n-norm corresponds to the Euclidean norm and the Schmidt vector 1-norm is the local analogue of the Euclidean norm. Similarly, the Schmidt operator n-norm equals the standard operator norm and the Schmidt operator 1-norm is exactly the local analogue of the operator norm. For $1 < k < n$, the Schmidt k-norms are in between the two extremes in a way that is made precise via the Schmidt rank of pure states. The Schmidt vector norms have recently appeared in [2, 3] as a tool for testing k-positivity of linear maps. The Schmidt operator and vector 2-norms have appeared in literature related to NPPT bound entangled states [4, 5]. Additionally, the Schmidt operator 1-norm of a normal operator coincides with the recently-explored *local numerical radius* [6].

We begin with a systematic study of the Schmidt norms. After deriving their basic properties, we focus on bounding them in a variety of ways and computing them in special cases. Once we have some strong tools built up for handling these norms, we show that several of the previously-known applications and results of these norms follow very simply. We explore a connection with k-positivity of linear maps and the existence of NPPT bound entangled states, as well as connections with quantum fidelity and trace distance, regularized relative entropy of entanglement [7, 8], and convex mapping cones [9, 10].

The paper is arranged as follows. In Section 2 we present our notation and terminology and introduce the reader to the required notions from operator theory and quantum information. In Section 3 we will define and explore Schmidt vector norms, which can be thought of as measuring how close pure states are to having a given Schmidt rank. We will then define Schmidt operator norms in Section 4, which have the same interpretation as the vector norms, but apply to arbitrary mixed states. We will see that the Schmidt operator norms are very difficult to calculate in general, so we will develop several inequalities to bound them in various situations.

Section 5 will focus on the problem of determining whether or not a given operator is k-positive – in the language of quantum information this is the problem of determining whether or not that operator is a k-entanglement witness. We will see that the Schmidt operator norms can be used to derive several testable conditions for k positivity, and we will derive a complete characterization in the case when the operator has two distinct eigenvalues. In Section 6 we will see that our interpretation of the Schmidt operator norms as a measure of how close a state is to having small Schmidt rank makes sense in terms of quantum fidelity. We will also connect the norms to a conjecture of Brandao [11] and show that our previously-derived bounds verify that the conjecture holds for fixed finite dimensions.

In Section 7 we will introduce the reader to semidefinite programming and derive a family of semidefinite programs that can be used to bound the Schmidt operator norms. We will examine some low-dimensional cases when the family of semidefinite programs collapses to a single program that can compute the norms within any desired accuracy, and the duality theory of semidefinite programs will lead to some theoretical results about the norms. In Section 8 we will define similar norms over arbitrary convex mapping cones and show that many of our results hold in a more general setting. In particular, we will define and explore a norm that measures how close a state is to having positive partial transpose.

2. Preliminaries

We will use $\mathcal{H}$ to denote a finite-dimensional complex Hilbert space and $\mathcal{L}(\mathcal{H})$ to denote the set of linear operators on $\mathcal{H}$. When the dimension of the Hilbert space is important, we will denote it $\mathcal{H}_n$, where n is its dimension. Similarly, id_n will represent the identity map on $\mathcal{L}(\mathcal{H}_n)$. Of particular interest in quantum information is the case when $\mathcal{H}$ is a *bipartite system* – a tensor product of two smaller Hilbert spaces $\mathcal{H} = \mathcal{H}_n \otimes \mathcal{H}_m$. We will assume for the sake of brevity throughout the paper that $m \leq n$. A vector $|v\rangle \in \mathcal{H}$ is denoted using Dirac bra-ket notation, with $\langle v| := |v\rangle^*$. Whenever we use this bra-ket notation, it will be assumed that $|v\rangle$ is such that $\big\||v\rangle\big\| = 1$ and so $|v\rangle$ represents a pure state (or more correctly the associated state is given by the rank one projection $|v\rangle\langle v|$). When it is not important that the vector under consideration be normalized, it will be written as a lowercase letter without the surrounding ket (such as v or w). We will denote the computational basis vectors (i.e., the vectors with 1 in the i^{th} component and 0 in all other components) by $\{|e_i\rangle\}$.

If $X \in \mathcal{L}(\mathcal{H})$ is positive then we will write $X \geq 0$ or $X \in \mathcal{L}(\mathcal{H})^+$. A (mixed) quantum state is represented by a *density operator* $\rho \geq 0$ that satisfies $\mathrm{Tr}(\rho) = 1$. Whenever lowercase Greek letters like ρ or σ are used, it is assumed that they are density operators. General operators will be represented by uppercase letters like X and Y.

Given a linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$, we can define its dual map $\Phi^\dagger : \mathcal{L}(\mathcal{H}_m) \rightarrow \mathcal{L}(\mathcal{H}_n)$ via the Hilbert-Schmidt inner product $\mathrm{Tr}(\Phi(X)Y) = \mathrm{Tr}(X\Phi^\dagger(Y))$. The map Φ is said to be:

- *Hermicity-preserving* if $\Phi(X)^* = \Phi(X)$ whenever $X^* = X$.
- *Positive* if $\Phi(X) \geq 0$ whenever $X \geq 0$.
- *k-positive* if $(id_k \otimes \Phi)(X) \geq 0$ whenever $X \in (\mathcal{L}(\mathcal{H}_k) \otimes \mathcal{L}(\mathcal{H}_n))^+$.
- *Completely positive* if Φ is k-positive for all $k \in \mathbb{N}$.

A theorem of Choi says that n-positivity of Φ is equivalent to complete positivity of Φ [12, 13]. Furthermore, Φ is completely positive if and only if $(id_n \otimes \Phi)(E) \geq 0$, where $E := \frac{1}{n}\sum_{i,j=1}^n |e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j|$. The matrix form for the operator $(id_n \otimes \Phi)(E)$ is referred to as the *Choi matrix* of Φ. In fact, the Choi matrix defines an isomorphism (known as the *Choi-Jamiolkowski isomorphism* [14]) between linear maps $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ and operators $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$. In keeping with the terminology of [10, 15], we will say that a Hermitian operator $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is *k-block positive* if the associated linear map is k-positive.

Several connections will be made between the Schmidt norms and other more well-known norms. In particular, for an operator $X \in \mathcal{L}(\mathcal{H}_n)$ it will be useful to be familiar with the *Ky Fan k-norm* [16] of X, given by $\big\|X\big\|_k := \sum_{i=1}^k s_i$, where $s_1 \geq \cdots \geq s_n$ are the singular values of X. Note that the smallest of the Ky Fan norms, the Ky Fan 1-norm, is equal to the operator norm. The largest of the Ky Fan norms, the Ky Fan n-norm, is equal to the *trace norm* because it can be written as $\big\|X\big\|_n = \mathrm{Tr}(|X|)$, where $|X| := \sqrt{X^*X}$ is the absolute value of X.

2.1. **Schmidt Rank and Schmidt Number.** The Schmidt Decomposition Theorem [1, Section 2.5] is a basic tool in quantum information theory. It states that if $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ then there exists $k \leq m$ (recall that $m \leq n$ by assumption) and orthonormal sets of vectors $\{|u_1\rangle, |u_2\rangle, \ldots, |u_k\rangle\} \subset \mathcal{H}_n$ and $\{|v_1\rangle, |v_2\rangle, \ldots, |v_k\rangle\} \subset \mathcal{H}_m$ such that

$$|v\rangle = \sum_{i=1}^k \alpha_i |u_i\rangle \otimes |v_i\rangle \tag{1}$$

for some non-negative real constants $\{\alpha_i\}$.

The standard proof of the Schmidt Decomposition works by noticing that there is an isomorphism between $\mathcal{H}_n \otimes \mathcal{H}_m$ and $\mathcal{L}(\mathcal{H}_n, \mathcal{H}_m)$ given by associating a vector $|u_i\rangle \otimes |v_i\rangle$ with the operator $|u_i\rangle\overline{\langle v_i|}$ and extending linearly. We will denote the operator associated with the vector $|v\rangle$ by A_v. Applying the singular value decomposition to A_v gives the Schmidt Decomposition of $|v\rangle$.

In the Schmidt Decomposition (1) of $|v\rangle$, the least number of terms required in the summation is known as the *Schmidt rank* of $|v\rangle$, denoted $SR(|v\rangle)$. It follows that the

Schmidt rank of $|v\rangle$ is equal to the number of non-zero singular values of the operator to which $|v\rangle$ is associated (i.e., its rank). Similarly, the α_i's are exactly the singular values of A_v. Because the singular value decomposition is easy to compute, so are the Schmidt rank and the Schmidt Decomposition of an arbitrary pure state $|v\rangle$.

One of the most useful non-trivial theorems about the Schmidt rank is the following result of Cubitt, Montanaro and Winter [17], which provides a tight bound on the dimension of subspaces consisting entirely of vectors with high Schmidt rank.

Theorem 2.1. *The maximum dimension of a subspace $\mathcal{S} \subseteq \mathcal{H}_n \otimes \mathcal{H}_m$ such that $SR(|v\rangle) \geq k$ for all $|v\rangle \in \mathcal{S}$ is given by $(n-k+1)(m-k+1)$.*

Not only is $(n-k+1)(m-k+1)$ shown to be an upper bound on the dimension of such subspaces, but an explicit method of construction is given that produces such a subspace that attains the bound.

In analogy with the Schmidt rank for pure states, the *Schmidt number* [18] of a mixed state ρ is defined to be the least natural number k such that ρ can be written as

$$\rho = \sum_i p_i |v_i\rangle\langle v_i|,$$

where $SR(|v_i\rangle) \leq k$ for all i and $\{p_i\}$ forms a probability distribution. The Schmidt number of a state can be thought of as a rough measure of how entangled that state is. One case that is of particular interest is when $SN(\rho) = 1$, in which case ρ is said to be *separable*. It is not difficult to check that ρ is separable if and only if it can be written as $\rho = \sum_i X_i \otimes Y_i$ for some $\big\{X_i\big\}, \big\{Y_i\big\} \geq 0$. Observe that the set of states ρ with $SN(\rho) \leq k$ is a closed convex cone if we remove the requirement that $\mathrm{Tr}(\rho) = 1$. This cone will be denoted $\mathcal{S}_k$.

Given a convex cone $\mathcal{C} \subseteq \mathcal{L}(\mathcal{H})$, its *dual cone* is the convex cone defined through the Hilbert-Schmidt inner product as follows:

$$\mathcal{C}^O := \big\{X \in \mathcal{L}(\mathcal{H}) : \mathrm{Tr}(XY) \geq 0 \quad \forall\, Y \in \mathcal{C}\big\}.$$

It is known that the dual cone of $\mathcal{S}_k$, the operators with Schmidt number no greater than k, is exactly the set of k-block positive operators, and vice-versa [15].

2.2. **Relationship Between Schmidt Number and k-Positivity.** The following two theorems of Terhal and Horodecki [18] show the intricate link between k-positive maps and states with Schmidt number not greater than k.

Theorem 2.2. *Let $\Phi : \mathcal{L}(\mathcal{H}_n) \to \mathcal{L}(\mathcal{H}_n)$ be a linear map. Then Φ is k-positive if and only if*

$$(id_n \otimes \Phi)(\rho) \geq 0 \quad \forall\, \rho \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n) \ \textit{with}\ SN(\rho) \leq k.$$

Theorem 2.3. *Let $\rho \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n)$ be a density operator. Then $SN(\rho) \leq k$ if and only if*

$$(id_n \otimes \Phi)(\rho) \geq 0 \quad \forall\, k\textit{-positive}\ \Phi : \mathcal{L}(\mathcal{H}_n) \to \mathcal{L}(\mathcal{H}_n).$$

The following two corollaries are easily proved in light of the previous two theorems; we state them explicitly as they will be of particular importance for us.

Corollary 2.4. *Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$. Then X is k-block positive if and only if*

$$\mathrm{Tr}(X\rho) \geq 0 \quad \forall\, \rho \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m) \text{ with } SN(\rho) \leq k.$$

Corollary 2.5. *Let $\rho \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ be a density operator. Then $SN(\rho) \leq k$ if and only if*

$$\mathrm{Tr}(X\rho) \geq 0 \quad \forall\, k\text{-positive } X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m).$$

These theorems are of theoretical interest, but are not of much practical use for testing k-positivity or Schmidt number, since (for example) it is not possible to apply $id_n \otimes \Phi_k$ to ρ and check positivity for every k-positive map Φ. In some cases, however, an explicit finite set $\mathcal{S}$ of k-positive maps is known for which $(id_n \otimes \Phi_k)(\rho) \geq 0$ for all $\Phi_k \in \mathcal{S}$ implies $SR(\rho) \leq k$. For example, if $m = 2$ and $n = 2$ or $n = 3$ then the transpose map T alone is enough to determine whether or not ρ is separable (i.e., $SN(\rho) = 1$) [19].

The fact that the transpose map can be used to determine separability in small dimensions has led to the study of *positive partial transpose (PPT)* states [20], which are density operators ρ such that $(id_n \otimes T)(\rho) \geq 0$. Throughout the rest of this paper, we will write the partial transpose operation $(id_n \otimes T)(\rho)$ as ρ^Γ.

The connections between k-positivity and Schmidt number have recently been linked with the dual cone relationship of k-positivity and Schmidt number [10, 15]. This basic theme of duality between k-positivity and Schmidt number will be present throughout much of this paper.

3. Schmidt Vector Norms

With the Schmidt Decomposition in hand, we can define a new family of norms that are analogous to the standard Euclidean norm. These norms were very recently considered independently in [2, 3] as a tool for detecting k-block positivity of operators.

Definition 3.1. *Let $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ and let $1 \leq k \leq m$. Then we define the* Schmidt vector norm *of $|v\rangle$, denoted $\big\||v\rangle\big\|_{s(k)}$, by*

$$\big\||v\rangle\big\|_{s(k)} := \sup_{|w\rangle} \Big\{ \big|\langle w|v\rangle\big| : SR(|w\rangle) \leq k \Big\}.$$

Note that even though this definition is only stated for unit vectors $|v\rangle$, it extends in the obvious way to a norm on all of $\mathcal{H}_n \otimes \mathcal{H}_m$. Some observations are now in order. First, the case of $k = m$ is very familiar: $\big\||v\rangle\big\|_{s(m)} = \big\||v\rangle\big\|$. Also, it is clear from the definition that one has $\big\||v\rangle\big\|_{s(k)} \leq \big\||v\rangle\big\|$ for all k. In fact, it is not difficult to see that we have an increasing family of norms leading up to the standard Euclidean norm:

$$\big\||v\rangle\big\|_{s(1)} \leq \big\||v\rangle\big\|_{s(2)} \leq \cdots \leq \big\||v\rangle\big\|_{s(m-1)} \leq \big\||v\rangle\big\|.$$

The first result of this chapter shows that this norm is actually not particularly difficult to calculate.

Theorem 3.2. *Let $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ have Schmidt coefficients $\{\alpha_i\}$. Then*

$$\big\||v\rangle\big\|_{s(k)} = \sqrt{\sum_{i=1}^{k} \alpha_i^2}.$$

Proof. To see that $\big\||v\rangle\big\|_{s(k)} \geq \sqrt{\sum_{i=1}^{k} \alpha_i^2}$, use the Schmidt Decomposition to write $|v\rangle = \sum_{i=1}^{m} \alpha_i |u_i\rangle \otimes |v_i\rangle$. Now let

$$|w\rangle = \frac{\sum_{i=1}^{k} \alpha_i |u_i\rangle \otimes |v_i\rangle}{\sqrt{\sum_{i=1}^{k} \alpha_i^2}}.$$

Observe that $SR(|w\rangle) \leq k$. Some algebra then reveals that

$$\begin{aligned}
\langle w|v\rangle &= \frac{1}{\sqrt{\sum_{i=1}^{k} \alpha_i^2}} \Big(\sum_{i=1}^{m} \alpha_i \langle u_i| \otimes \langle v_i| \Big) \Big(\sum_{i=1}^{k} \alpha_i |u_i\rangle \otimes |v_i\rangle \Big) \\
&= \frac{1}{\sqrt{\sum_{i=1}^{k} \alpha_i^2}} \sum_{i=1}^{m} \sum_{j=1}^{k} \alpha_i \alpha_j \langle u_i|u_j\rangle \otimes \langle v_i|v_j\rangle \\
&= \frac{1}{\sqrt{\sum_{i=1}^{k} \alpha_i^2}} \sum_{j=1}^{k} \alpha_j^2 \\
&= \sqrt{\sum_{i=1}^{k} \alpha_i^2}.
\end{aligned}$$

To see the opposite inequality, consider some fixed $|w\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ with $SR(|w\rangle) \leq k$ and Schmidt Decomposition $|w\rangle = \sum_{i=1}^{k} \beta_i |w_i\rangle \otimes |x_i\rangle$. Then

$$\big|\langle w|v\rangle\big| = \left| \Big(\sum_{i=1}^{m} \alpha_i \langle u_i| \otimes \langle v_i| \Big) \Big(\sum_{i=1}^{k} \beta_i |w_i\rangle \otimes |x_i\rangle \Big) \right| \leq \sum_{i=1}^{m} \sum_{j=1}^{k} \alpha_i \beta_j \big|\langle u_i|w_j\rangle \langle v_i|x_j\rangle\big| = \alpha^* D \beta$$

where $\alpha^T = (\alpha_1, \cdots, \alpha_m)$ and $\beta^T = (\beta_1, \cdots, \beta_k, 0, \cdots, 0)$ are vectors of Schmidt coefficients, and D is the matrix given by $D_{ij} = \big|\langle u_i|w_j\rangle\langle v_i|x_j\rangle\big|$ in which we have extended $\{|u_i\rangle\}$, $\{|w_j\rangle\}$ and $\{|x_j\rangle\}$ to orthonormal bases of their respective spaces. Observe that D is doubly-sub-stochastic (i.e., each of its row and column sums is no greater than 1) so the Hardy-Littlewood-Polya Theorem tells us that the vector $\gamma := D\beta$ satisfies

$$\sum_{i=1}^{j} \gamma_i \leq \sum_{i=1}^{j} \beta_i \quad \forall\, 1 \leq j \leq m.$$

It follows from some simple linear algebra and the Cauchy-Schwarz inequality that

$$\alpha^* D \beta \leq \alpha^* \beta \leq \sqrt{\sum_{i=1}^{k} \alpha_i^2} \sqrt{\sum_{i=1}^{k} \beta_i^2} = \sqrt{\sum_{i=1}^{k} \alpha_i^2}.$$

□

One particularly useful way of looking at Theorem 3.2 is to notice that, because the Schmidt coefficients of $|v\rangle$ are the singular values of the operator A_v to which $|v\rangle$ is associated in the proof of the Schmidt Decomposition Theorem, it follows that $\big\||v\rangle\big\|_{s(k)}^2 = \big\|A_v^* A_v\big\|_k$.

The following corollary of Theorem 3.2 is easy to check and is thus presented without proof.

Corollary 3.3. *Let $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ and suppose $h \leq k$. Then*

$$\big\||v\rangle\big\|_{s(h)} \leq \big\||v\rangle\big\|_{s(k)} \leq \sqrt{\frac{k}{h}}\big\||v\rangle\big\|_{s(h)}.$$

Furthermore, equality is achieved on the left if and only if $\big\||v\rangle\big\|_{s(h)} = 1$ if and only if $SR(|v\rangle) \leq h$. Equality is achieved on the right if and only if the k largest Schmidt coefficients of $|v\rangle$ are equal.

4. Schmidt Operator Norms

It is natural now to define a family of norms for operators that is analogous to the family of Schmidt vector norms.

Definition 4.1. *Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ and $1 \leq k \leq m$. Then we define the* Schmidt operator norms *of X, denoted $\big\|X\big\|_{S(k)}$, by*

$$\big\|X\big\|_{S(k)} := \sup_{|v\rangle,|w\rangle} \Big\{ \big|\langle w|X|v\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k \Big\}.$$

Some minor observations are that $\big\|X\big\|_{S(m)} = \big\|X\big\|$ and $\big\|X\big\|_{S(k)} \leq \big\|X\big\|$ for all k. Further, in analogy with the Schmidt vector norms, the Schmidt operator norms form an increasing family of norms that lead up to the standard operator norm:

$$\big\|X\big\|_{S(1)} \leq \big\|X\big\|_{S(2)} \leq \cdots \leq \big\|X\big\|_{S(m-1)} \leq \big\|X\big\|.$$

Moreover, although $\big\|X^*\big\|_{S(k)} = \big\|X\big\|_{S(k)}$, it is *not* the case in general that $\big\|X^*X\big\|_{S(k)} = \big\|X\big\|_{S(k)}^2$. They also do not satisfy any natural submultiplicativity relationships.

We will see that computing the Schmidt operator norms is very difficult in general. The following result shows that we can nonetheless exactly compute the Schmidt operator norms when the operator under consideration has rank 1. The proof follows simply from the relevant definitions and hence we leave it to the interested reader.

Proposition 4.2. *Let $X = |w\rangle\langle v| \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ be a rank-1 operator. Then*

$$\big\|X\big\|_{S(k)} = \big\||w\rangle\big\|_{s(k)}\big\||v\rangle\big\|_{s(k)}.$$

We now present an important example to make use of Proposition 4.2.

Example 4.3. Recall the rank-1 projection operator $E = \frac{1}{n}\sum_{i,j=1}^{n} |e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j| \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n)$. By Proposition 4.2 we have that

$$\big\|E\big\|_{S(k)} = \big\| \sum_{i=1}^{n} \frac{1}{\sqrt{n}} |e_i\rangle \otimes |e_i\rangle \big\| = \sum_{i=1}^{k} \left(\frac{1}{\sqrt{n}}\right)^2 = \frac{k}{n}.$$

We will see that this simple example can be used to show that some inequalities that we derive in the next section are tight. It will also have applications to bound entanglement in Section 5.1.

The following proposition shows if X is normal then it is enough to take the supremum only over $|v\rangle$ in the definition of the Schmidt operator norms.

Proposition 4.4. *Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ be normal. Then*

$$\big\|X\big\|_{S(k)} = \sup_{|v\rangle} \Big\{ \big|\langle v|X|v\rangle\big| : SR(|v\rangle) \leq k \Big\}.$$

Proof. Write X in its Spectral Decomposition as $X = \sum_i \lambda_i |v_i\rangle\langle v_i|$. Then observe that the set of states $|v\rangle$ with $SR(|v\rangle) \leq k$ is compact, hence we can find a particular $|v\rangle$ with $SR(|v\rangle) \leq k$ such that $\sup_{|v\rangle}\{|\langle v|X|v\rangle| : SR(|v\rangle) \leq k\} = \sum_i |\lambda_i||\langle v_i|v\rangle|^2$. Then for any $|w\rangle$ with $SR(|w\rangle) \leq k$, we have that $|\langle w|X|w\rangle| = \sum_i |\lambda_i||\langle v_i|w\rangle|^2 \leq \sum_i |\lambda_i||\langle v_i|v\rangle|^2$. Now define the i^{th} component of two vectors v' and w' by $v'_i := \sqrt{|\lambda_i|}|\langle v_i|v\rangle|$ and $w'_i := \sqrt{|\lambda_i|}|\langle w|v_i\rangle|$. Simply applying the Cauchy-Schwarz inequality to v' and w' gives $|\langle w|X|v\rangle| \leq |\langle v|X|v\rangle|$. The other inequality is trivial. □

This result captures a well-known property of the operator norm in the $k = m$ case. We also note that Proposition 4.4 says that the Schmidt 1-norm, $\big\|\cdot\big\|_{S(1)}$, when acting on normal operators, coincides with the *local spectral radius* r^{loc} [6]. That is, if X is normal then $\big\|X\big\|_{S(1)} = r^{loc}(X)$.

The following corollary shows perhaps a more natural way of looking at $\big\|X\big\|_{S(k)}$ from the quantum information perspective.

Corollary 4.5. *Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ be normal. Then*

$$\big\|X\big\|_{S(k)} = \sup_{\rho} \Big\{ |\mathrm{Tr}(X\rho)| : SN(\rho) \leq k \Big\}.$$

Proof. Proposition 4.4 says that

$$\big\|X\big\|_{S(k)} = \sup_{|v\rangle} \Big\{ \big|\langle v|X|v\rangle\big| : SR(|v\rangle) \leq k \Big\} = \sup_{|v\rangle} \Big\{ |\mathrm{Tr}(X|v\rangle\langle v|)| : SR(|v\rangle) \leq k \Big\}.$$

The maximum on the right cannot become larger when taking the supremum over mixed states since a mixed state can be written as a convex combination of pure states. □

The following corollary shows that the Schmidt operator norms are non-increasing under local quantum operations.

Corollary 4.6. *Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ be normal and let $\Phi : \mathcal{L}(\mathcal{H}_m) \to \mathcal{L}(\mathcal{H}_m)$ be a completely positive map. Then*

$$\big\|(id_n \otimes \Phi)(X)\big\|_{S(k)} \leq \big\|X\big\|_{S(k)}.$$

Proof. By Corollary 4.5 we know that

$$\begin{aligned} \big\|(id_n \otimes \Phi)(X)\big\|_{S(k)} &= \sup_{\rho} \Big\{ |\mathrm{Tr}((id_n \otimes \Phi)(X)\rho)| : SN(\rho) \leq k \Big\} \\ &= \sup_{\rho} \Big\{ |\mathrm{Tr}(X(id_n \otimes \Phi^\dagger)(\rho))| : SN(\rho) \leq k \Big\}. \end{aligned}$$

The result follows from the fact that Schmidt number is non-increasing under the action of local CP maps [18], so $SN((id_n \otimes \Phi^\dagger)(\rho)) \leq k$. □

Finally, the last result of this section shows that there is an intimate connection between the Schmidt operator norms and k-block positivity of an operator.

Corollary 4.7. *Let $X \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$ be positive and let $c \in \mathbb{R}$. Then $cI - X$ is k-block positive if and only if $c \geq \big\|X\big\|_{S(k)}$.*

Proof. By Corollary 2.4 we know that $cI - X$ is k-block positive if and only if

$$\mathrm{Tr}((cI - X)\rho) = c - \mathrm{Tr}(X\rho) \geq 0 \quad \forall\, \rho \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m) \text{ with } SN(\rho) \leq k.$$

Corollary 4.5 tells us that this is true precisely when $c \geq \big\|X\big\|_{S(k)}$. □

In particular, Corollary 4.7 shows that the problem of computing Schmidt operator norms is equivalent to the problem of determining k-block positivity of a Hermitian operator. Since the k-positivity problem is very difficult in general, computing these norms even just for positive operators must be a very difficult problem as well.

4.1. **Schmidt Operator Norm Inequalities.** Since computing the Schmidt operator norms in general is quite difficult, it will be useful to have explicitly calculable bounds for them. The following upper bound is thus of interest because it is explicitly computable in light of Theorem 3.2.

Proposition 4.8. *Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ be normal with eigenvalues $\{\lambda_i\}$ and corresponding eigenvectors $\{|v_i\rangle\}$. Then*

$$\big\|X\big\|_{S(k)} \leq \sum_i |\lambda_i| \big\||v_i\rangle\big\|_{s(k)}^2.$$

Proof. Let $|v\rangle$ and $|w\rangle$ have $SR(|v\rangle), SR(|w\rangle) \leq k$. Then we have

$$\big|\langle w|X|v\rangle\big| = \big|\sum_i \lambda_i \langle w|v_i\rangle\langle v_i|v\rangle\big| \leq \sum_i |\lambda_i||\langle w|v_i\rangle||\langle v_i|v\rangle| \leq \sum_i |\lambda_i| \big\||v_i\rangle\big\|_{s(k)}^2.$$

□

Because $\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is finite-dimensional, we know that the Schmidt operator norms must be equivalent. In order to quantify this fact, we will first need a simple lemma.

Lemma 4.9. *Let $h \leq k$ and suppose $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ is a unit vector with $SR(|v\rangle) \leq k$. Then there exist nonnegative real constants $\{d_j\}$ and (not necessarily distinct) unit vectors $\{|v_j\rangle\} \subseteq \mathcal{H}_n \otimes \mathcal{H}_m$ for $1 \leq j \leq k$ such that $\sum_{j=1}^k d_j^2 = h$, $SR(|v_j\rangle) \leq h$, and*

$$h|v\rangle = \sum_{j=1}^k d_j |v_j\rangle.$$

Proof. We can write $|v\rangle$ via the Schmidt Decomposition as $|v\rangle = \sum_{j=1}^k c_j |a_j\rangle \otimes |b_j\rangle$ with $\sum_{j=1}^k |c_j|^2 = 1$ and $\{|a_j\rangle\}$, $\{|b_j\rangle\}$ orthonormal sets. Thus

$$h|v\rangle = \sum_{i=1}^h \sum_{j=1}^k c_j |a_j\rangle \otimes |b_j\rangle.$$

Because $h \leq k$, we can rearrange the summations in such a way that we sum over k sets of orthonormal vectors, with h vectors in each set. We thus have $h|v\rangle = \sum_{j=1}^{k} d_j|v_j\rangle$ for some unit vectors $|v_j\rangle$ with $SR(|v_j\rangle) \leq h$ and constants d_j satisfying $\sum_{j=1}^{k} d_j^2 = h$. □

Theorem 4.10. *Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ and suppose $h \leq k$. Then*

$$\big\|X\big\|_{S(h)} \leq \big\|X\big\|_{S(k)} \leq \frac{k}{h}\big\|X\big\|_{S(h)}.$$

Proof. The left inequality is trivial by the definition of the Schmidt operator norms. To see the right inequality, suppose $|v\rangle$ and $|w\rangle$ have $SR(|v\rangle), SR(|w\rangle) \leq k$. Use Lemma 4.9 to write $h|v\rangle = \sum_{j=1}^{k} d_j|v_j\rangle$ and $h|w\rangle = \sum_{j=1}^{k} f_j|w_j\rangle$ so that

$$h^2\big|\langle w|X|v\rangle\big| = \big|\sum_{i,j=1}^{k} f_i d_j \langle w_i|X|v_j\rangle\big| \leq \Big(\sum_{i=1}^{k} f_i\Big)\Big(\sum_{i=1}^{k} d_i\Big)\big\|X\big\|_{S(h)} \leq kh\big\|X\big\|_{S(h)},$$

where the rightmost inequality follows from two applications of the Cauchy-Schwarz inequality. The result follows by dividing through by h^2. □

To see that the inequalities of Theorem 4.10 are tight, simply recall Example 4.3. Also observe that a straightforward consequence of this result is the inequality $\big\|X\big\|_{S(k)} \geq \frac{k}{m}\big\|X\big\|$ for all $k \leq m$. The following results provide lower bounds that are much better in many situations.

Proposition 4.11. *Let $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ have eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{mn}$. Then $\big\|X\big\|_{S(k)} \geq \lambda_{nm-(n-k)(m-k)}$. Furthermore, there exists an $X \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$ such that $\big\|X\big\|_{S(k)} < \lambda_{nm-(n-k)(m-k)+1}$.*

Proof. Let $\mathcal{V}$ be the span of the eigenvectors $|v_{nm-(n-k)(m-k)}\rangle, |v_{nm-(n-k)(m-k)+1}\rangle, \ldots, |v_{nm}\rangle$ corresponding to $\lambda_{nm-(n-k)(m-k)}, \lambda_{nm-(n-k)(m-k)+1}, \ldots, \lambda_{mn}$. Then because $\dim(\mathcal{V}) = (n-k)(m-k)+1$, by Theorem 2.1, we know that there must exist a vector $|v\rangle \in \mathcal{V}$ with $SR(|v\rangle) \leq k$. It follows that

$$\big|\langle v|X|v\rangle\big| \geq \sum_{i=nm-(n-k)(m-k)}^{mn} \lambda_i\big|\langle v_i|v\rangle\big|^2 \geq \lambda_{nm-(n-k)(m-k)}.$$

To see the final claim, note that the dimension given by Theorem 2.1 is tight, so we can construct a positive operator X with distinct eigenvalues such that the span of the eigenvectors corresponding to its $(n-k)(m-k)$ largest eigenvalues does not contain any states $|w\rangle$ with $SR(|w\rangle) \leq k$. It follows that $\langle v|X|v\rangle < \lambda_{nm-(n-k)(m-k)+1}$ for all $|v\rangle$ with $SR(|v\rangle) \leq k$. □

Corollary 4.12. *Let $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ have eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{mn}$. Then for any $r \geq k$,*

$$\big\|X\big\|_{S(k)} \geq \frac{k\lambda_{mn-(n-r)(m-r)}}{r}.$$

Proof. By Proposition 4.11 we have that $\big\|X\big\|_{S(r)} \geq \lambda_{mn-(n-r)(m-r)}$. Using Theorem 4.10 shows that if $k \leq r$ then

$$\big\|X\big\|_{S(k)} \geq \frac{k}{r}\big\|X\big\|_{S(r)} \geq \frac{k\lambda_{mn-(n-r)(m-r)}}{r}.$$

□

Now notice that if $P = P^* = P^2 \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$ is an orthogonal projection, then by Theorem 4.10 we have that $\frac{k}{m} \leq \big\|P\big\|_{S(k)} \leq 1$. The left inequality was seen to be tight by a rank-1 projection in Example 4.3, and it is not difficult to construct projection operators of any rank that have $\big\|P\big\|_{S(k)} = 1$. However, the following two results show that we can improve the lower bound if we take $\mathrm{rank}(P)$ into account.

Corollary 4.13. *Let $P = P^* = P^2 \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$ be an orthogonal projection. Then*

$$\big\|P\big\|_{S(k)} \geq \min\Big\{1, \frac{k}{\lceil\frac{1}{2}\big(n+m-\sqrt{(n-m)^2+4\mathrm{rank}(P)-4}\big)\rceil}\Big\}.$$

Proof. Notice that Proposition 4.11 implies that $\big\|P\big\|_{S(r)} = 1$ whenever $\mathrm{rank}(P) \geq (n-r)(m-r)+1$. Solving this inequality for r gives

$$r \geq \frac{1}{2}\Big(n+m-\sqrt{(n-m)^2+4\mathrm{rank}(P)-4}\Big).$$

Thus, choose $r = \Big\lceil\frac{1}{2}\Big(n+m-\sqrt{(n-m)^2+4\mathrm{rank}(P)-4}\Big)\Big\rceil$ and $k \leq r$. Then by Corollary 4.12 we have

$$\big\|P\big\|_{S(k)} \geq \frac{k}{\lceil\frac{1}{2}\big(n+m-\sqrt{(n-m)^2+4\mathrm{rank}(P)-4}\big)\rceil}.$$

□

Corollary 4.13 is particularly important because we will see that several important problems in quantum information theory could be answered if we were able to compute, or bound tightly, the Schmidt operator norms of projections. It is the best bound we have when $\mathrm{rank}(P)$ is small or large (e.g., $\mathrm{rank}(P) \leq m$ or $\mathrm{rank}(P) \geq (n-1)(m-1)$), but the following theorem shows that we can do much better for moderate-rank projections (e.g., when $\mathrm{rank}(P) \approx \frac{mn}{2}$).

Theorem 4.14. *Let $P = P^* = P^2 \in (\mathcal{L}(\mathcal{H}_n)\otimes\mathcal{L}(\mathcal{H}_m))^+$ be an orthogonal projection. Then*

$$\big\|P\big\|_{S(k)} \geq \frac{(k-1)mn+(m-k)\mathrm{rank}(P)}{mn(m-1)}.$$

Proof. We first prove the result in the $k = 1$ case. Define $r := \mathrm{rank}(P)$. Choose orthonormal bases $\{|e_j\rangle\}$ and $\{|f_l\rangle\}$ of $\mathcal{L}(\mathcal{H}_n)$ and $\mathcal{L}(\mathcal{H}_m)$, respectively. Then choose r orthonormal vectors $|v_i\rangle$ in the range of P and observe that we can write them in the form

$$|v_i\rangle = \sum_{j=1}^{n}\sum_{l=1}^{m} c_{ijl}|e_j\rangle \otimes |f_l\rangle,$$

where $\{c_{ijl}\} \in \mathbb{C}$ is a family of constants such that

$$\sum_{j=1}^{n}\sum_{l=1}^{m}|c_{ijl}|^2 = 1 \quad \forall\, i = 1, 2, \ldots, r. \tag{2}$$

It follows that there exists some fixed j and l such that

$$\sum_{i=1}^{r}|c_{ijl}|^2 \geq \frac{r}{mn},$$

since otherwise Equation (2) would be violated. The $k = 1$ case follows by noting that, for this specific j and l,

$$\big\|P\big\|_{S(1)} \geq (\langle e_j| \otimes \langle f_l|)P(|e_j\rangle \otimes |f_l\rangle) = \sum_{i=1}^{r}\big|\langle v_i|(|e_j\rangle \otimes |f_l\rangle)\big|^2 = \sum_{i=1}^{r}|c_{ijl}|^2 \geq \frac{r}{mn}. \tag{3}$$

Now note that Theorem 3.2 says that for any $|v\rangle$ with $SR(|v\rangle) = 1$ and any $|w\rangle \in P\mathcal{H}$, $\big\|P\big\|_{S(1)} \geq \big|\langle v|w\rangle\big|^2 \geq \alpha_1^2$, where α_1 is the largest Schmidt coefficient of $|w\rangle$. On the other hand, it is clear that for any $|w\rangle \in P\mathcal{H}$, there exists a $|v\rangle$ with $SR(|v\rangle) = 1$ such that $\big|\langle v|w\rangle\big|^2 = \alpha_1^2$. It follows that

$$\big\|P\big\|_{S(1)} = \sup_{|w\rangle \in P\mathcal{H}}\big\{\alpha_1^2 : \alpha_1 \text{ is the largest Schmidt coefficient of } |w\rangle\big\}. \tag{4}$$

Now let $|w\rangle \in P\mathcal{H}$ have Schmidt coefficients $\{\alpha_i\}$ such that $\alpha_1 = \sqrt{\big\|P\big\|_{S(1)}}$. Then using the facts that $\sum_{i=1}^{m}\alpha_i^2 = 1$ and $\alpha_i \geq \alpha_j$ for $i \leq j$, it follows that $\sum_{i=2}^{m}\alpha_i^2 = 1 - \big\|P\big\|_{S(1)}$ and so $\sum_{i=2}^{k}\alpha_i^2 \geq \frac{k-1}{m-1}(1 - \big\|P\big\|_{S(1)})$. Thus

$$\big\|P\big\|_{S(k)} \geq \sum_{i=1}^{k}\alpha_i^2 = \big\|P\big\|_{S(1)} + \sum_{i=2}^{k}\alpha_i^2 \geq \big\|P\big\|_{S(1)} + \frac{(k-1)(1 - \big\|P\big\|_{S(1)})}{m-1}.$$

The result follows by rearranging and using Inequality (3). □

The two special cases of $k = 1$ and $k = m$ of Theorem 4.14 give lower bounds of $\frac{\mathrm{rank}(P)}{mn}$ and 1, respectively – the remaining lower bounds are just the linear interpolation of these two extremal cases. A comparison of the bounds provided by Corollary 4.13 and Theorem 4.14 is provided later in Figure 1.

Additionally, the same method as was used in the second half of the proof of Theorem 4.14 can be used to show the following improvement of the left inequality of Theorem 4.10 in the case of projections.

Corollary 4.15. *Let $P = P^* = P^2 \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$ be an orthogonal projection and let $h \leq k$. Then*

$$\big\|P\big\|_{S(k)} \geq \left(1 - \frac{k-h}{m-1}\right)\big\|P\big\|_{S(h)} + \frac{k-h}{m-1}.$$

5. Spectral Inequalities and Entanglement Witnesses

In this section we derive a set of conditions for testing when a Hermitian operator is and is not k-block positive. These tests are in the same flavour as the recently-developed tests of Chruściński and Kossakowski [2], which were stated in terms of the Ky Fan norm of the Kraus operators for the corresponding linear map. While their tests are explicitly computable, our tests are more theoretical, as they rely on computing the Schmidt operator norm on non-trivial projection operators. On the other hand, we show that they are in some ways much more powerful, as they imply not only the results of [2], but also several well-known tests for k-positivity that are *not* implied by the results of [2]. Our result also leads to a complete characterization of k-block positivity of Hermitian operators with exactly two distinct eigenvalues.

Furthermore, in [3] it was shown that Chruściński and Kossakowski's k-positivity tests could not be used to find entanglement witnesses that detect non-positive partial transpose states, and thus are not useful for trying to determine whether NPPT bound entangled states exist. We will see that our tests *are* able to find witnesses that detect such states.

Throughout this section, if $X = X^*$ then we will denote the positive eigenvalues of X by $\{\lambda_i^+\}$ and the corresponding eigenvectors by $\{|v_i^+\rangle\}$. We will similarly denote the negative eigenvalues by $\{\lambda_i^-\}$ and the corresponding eigenvectors by $\{|v_i^-\rangle\}$, and the eigenvectors corresponding to the zero eigenvalues by $\{|v_i^0\rangle\}$. $X^+ := \sum_i \lambda_i^+ |v_i^+\rangle\langle v_i^+| \geq 0$ and $X^- := \sum_i \lambda_i^- |v_i^-\rangle\langle v_i^-| \leq 0$ are defined to be the positive and negative parts of X, respectively. Similarly, $P_X^0 := \sum_i |v_i^0\rangle\langle v_i^0|$ and $P_X^- := \sum_i |v_i^-\rangle\langle v_i^-|$ denote the projections onto the nullspace and negative part of X, respectively.

Theorem 5.1. *Let $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$. Then*

(1) If $\big\|P_X^-\big\|_{S(k)} = 1$ then X is not k-block positive.

(2) If $\big\|P_X^0 + P_X^-\big\|_{S(k)} < 1$ and $\lambda_i^+ \geq \frac{\|X^-\|_{S(k)}}{1-\|P_X^0+P_X^-\|_{S(k)}}$ for all i, then X is k-block positive.

(3) If $\big\|P_X^-\big\|_{S(k)} < 1$, all of the negative eigenvalues are equal, X is nonsingular, and $\lambda_i^+ < \frac{\|X^-\|_{S(k)}}{1-\|P_X^-\|_{S(k)}}$ for all i, then X is not k-block positive.

Proof. To see statement (1), observe that there must be a vector $|v\rangle \in \mathrm{Range}(P_X^-)$ such that $SR(|v\rangle) \leq k$. It follows that $\langle v|X|v\rangle = \langle v|X^-|v\rangle < 0$ and so X is not k-block positive by Corollary 2.4.

To see statement (2), let $|v\rangle$ be such that $SR(|v\rangle) \leq k$ and define $\mu := \frac{\|X^-\|_{S(k)}}{1-\|P_X^0+P_X^-\|_{S(k)}}$. Then

$$\begin{aligned}\langle v|X|v\rangle = \langle v|X^+|v\rangle - \big|\langle v|X^-|v\rangle\big| &\geq \sum_i \lambda_i^+ |\langle v|v_i^+\rangle|^2 - \big\|X^-\big\|_{S(k)} \\ &\geq \mu \sum_i |\langle v|v_i^+\rangle|^2 - \big\|X^-\big\|_{S(k)} \geq \mu(1 - \|P_X^0 + P_X^-\|_{S(k)}) - \big\|X^-\big\|_{S(k)} = 0.\end{aligned}$$

To see statement (3), observe that the set of unit vectors $|v\rangle$ with $SR(|v\rangle) \leq k$ is compact and so there exists a particular $|v\rangle$ with $SR(|v\rangle) \leq k$ such that $\big|\langle v|X^-|v\rangle\big| = \big\|X^-\big\|_{S(k)}$.

Define $\mu := \frac{\|X^-\|_{S(k)}}{1-\|P_X^-\|_{S(k)}}$. Then

$$\langle v|X|v\rangle = \langle v|X^+|v\rangle - \big|\langle v|X^-|v\rangle\big| = \sum_i \lambda_i^+ |\langle v|v_i^+\rangle|^2 - \big\|X^-\big\|_{S(k)}$$
$$< \mu \sum_i |\langle v|v_i^+\rangle|^2 - \big\|X^-\big\|_{S(k)} = \mu(1 - \|P_X^-\|_{S(k)}) - \big\|X^-\big\|_{S(k)} = 0.$$

□

Even though Theorem 5.1 appears to be quite a technical result that may not be of much use due to the difficulty of computing the Schmidt k-norms, it is not difficult to derive computable corollaries from it. In fact, it implies a number of previously-known tests for k-positivity, as well as some new ones. We present some well-known tests here as corollaries for illustrative purposes. Other testable conditions can similarly be obtained by combining the inequalities of Theorem 5.1 with the inequalities derived in Section 4.1.

To see that Theorem 5.1 implies the k-positivity results of Chruściński and Kossakowski [2], use Proposition 4.8 and simply note that their usage of the Ky Fan norm of Kraus operators coincides with the Schmidt k-norm of the corresponding eigenvectors. It follows that this result also implies the k-positivity test of Takesaki and Tomiyama [21] and the positivity test of Benatti, Floreanini, and Piani [22]. Another corollary of this theorem is the following simple result of Kuah and Sudarshan [23].

Corollary 5.2. *Suppose $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ is a Hermicity-preserving linear map represented in its canonical Kraus representation $\Phi(\rho) = \sum_i \lambda_i^+ E_i \rho E_i^* + \sum_i \lambda_i^- F_i \rho F_i^*$, with the set of operators $\big\{E_1, E_2, \ldots, F_1, F_2, \ldots\big\}$ forming an orthonormal basis in the Hilbert-Schmidt inner product. If $\mathrm{rank}(F_i) \leq k$ for some i, then Φ is not k-positive.*

Proof. Simply recall that the Kraus operators E_i and F_i are exactly the operators to which the positive and negative eigenvectors of $X := (id_n \otimes \Phi)(E)$ are associated via the isomorphism used in the proof of the Schmidt Decomposition Theorem. Thus the rank of F_i coincides with the Schmidt rank of the corresponding eigenvector $|v_i\rangle$.

If $SR(|v_i\rangle) \leq k$ (i.e., $\mathrm{rank}(F_i) \leq k$) for some i then $\big|\langle v_i|P_X^-|v_i\rangle\big| = 1$ and so $\big\|P_X^-\big\|_{S(k)} = 1$. Condition (1) of Theorem 5.1 then gives the result. □

The following corollary provides a well-known characterization of the maximum number of negative eigenvalues that a k-block positive operator can have.

Corollary 5.3. *Suppose $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is k-block positive. Then it has at most $(n-k)(m-k)$ negative eigenvalues.*

Proof. Suppose X has more than $(n-k)(m-k)$ negative eigenvalues. Then, by Theorem 2.1 it follows that there exists $|v\rangle \in \mathrm{Range}(P_X^-)$ with $SR(|v\rangle) \leq k$. Hence we have $\big\|P_X^-\big\|_{S(k)} = 1$ and so condition (1) of Theorem 5.1 tells us that X is not k-block positive. □

The next corollary shows just how negative the negative eigenvalues of a k-block positive operator can be, and is also well-known.

Corollary 5.4. *Suppose $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is k-block positive. Denote the maximal and minimal eigenvalues of X by λ_{max} and λ_{min}, respectively. Then*

$$\frac{\lambda_{min}}{\lambda_{max}} \geq 1 - \frac{m}{k}.$$

Proof. If $\lambda_{min} \geq 0$ then the result is trivial. We thus assume that $\lambda_{min} < 0$. Suppose without loss of generality that X has only one negative eigenvalue and is nonsingular (if this is not the case, we can add a suitable positive operator Q to X so that $X + Q$ is k-block positive, has a single negative eigenvalue equal to λ_{min} and is nonsingular). If X is k-block positive then condition (1) of Theorem 5.1 says that $\left\|P_X^-\right\|_{S(k)} < 1$. Condition (3) then says that

$$\lambda_{max} \geq \frac{\|X^-\|_{S(k)}}{1 - \|P_X^-\|_{S(k)}} = -\lambda_{min} \frac{\|P_X^-\|_{S(k)}}{1 - \|P_X^-\|_{S(k)}}.$$

Then

$$\frac{\lambda_{min}}{\lambda_{max}} \geq \frac{\|P_X^-\|_{S(k)} - 1}{\|P_X^-\|_{S(k)}} = 1 - \frac{1}{\|P_X^-\|_{S(k)}} \geq 1 - \frac{m}{k}.$$

□

In fact, by using Corollary 4.13 and Theorem 4.14 in the final step of the above proof, we can derive the following bounds that in some sense "interpolate" between Corollary 5.3 and Corollary 5.4 by giving lower bounds on λ_{min} that depend on the number of negative eigenvalues of X.

Corollary 5.5. *Suppose $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is k-block positive with r negative eigenvalues. Denote the maximal and minimal eigenvalues of X by λ_{max} and λ_{min}, respectively. Then*

$$\frac{\lambda_{min}}{\lambda_{max}} \geq 1 - \frac{\left\lceil \frac{1}{2}\left(n + m - \sqrt{(n-m)^2 + 4r - 4}\right) \right\rceil}{k} \quad \text{and}$$
$$\frac{\lambda_{min}}{\lambda_{max}} \geq 1 - \frac{mn(m-1)}{(k-1)mn + (m-k)r}.$$

One final corollary shows that we now have a complete spectral characterization of the k-block positivity of Hermitian operators with exactly two distinct eigenvalues. The classification is trivial when both of the eigenvalues are negative or positive, but we believe that this is the first spectral classification for the case when they are of opposite signs.

Corollary 5.6. *Let $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ have two distinct eigenvalues $\lambda_1 > \lambda_2$. Then X is k-block positive if and only if*

$$\|P_X^-\|_{S(k)} \leq \frac{\lambda_1}{\lambda_1 - \lambda_2}. \tag{5}$$

Proof. If λ_1 and λ_2 have the same sign then the result is trivial. We thus assume that $\lambda_1 > 0$ and $\lambda_2 < 0$.

If X is k-block positive, then by condition (1) of Theorem 5.1 we know that $\|P_X^-\|_{S(k)} < 1$. Then condition (3) says that

$$\lambda_1 \geq \frac{\|X^-\|_{S(k)}}{1 - \|P_X^-\|_{S(k)}} = -\lambda_2 \frac{\|P_X^-\|_{S(k)}}{1 - \|P_X^-\|_{S(k)}}.$$

The desired inequality follows easily. To see the other direction of the proof, suppose inequality (5) is satisfied. Then because $\lambda_2 < 0$ it follows that $\|P_X^-\|_{S(k)} < 1$. $P_X^0 = 0$ by hypothesis, so simple algebra shows that condition (2) of Theorem 5.1 is satisfied. □

5.1. **Bound Entanglement and Werner States.** One of the most pressing open questions in quantum information theory is to find a classification of bound entangled states; that is, states with zero distillable entanglement. If a state is separable then it is bound entangled, as is any state with positive partial transpose. However, it is unknown whether or not there exist states with non-positive partial transpose (NPPT) that are bound entangled. It has been shown that a state is NPPT bound entangled if and only if $(\rho^\Gamma)^{\otimes k}$ is 2-positive for all $k \geq 1$ [5]. One especially important class of states in the study of bound entangled states is the family of Werner states [24], which can be parametrized by a single real variable $\alpha \in [-1, 1]$ via

$$\rho_\alpha := \frac{1}{n^2 - \alpha n}(I - \alpha E^\Gamma) \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n).$$

In particular, it is known that NPPT bound entangled states exist if and only if there is a Werner state that is NPPT bound entangled [25]. Because the partial transpose of Werner states have only two distinct eigenvalues, Corollary 5.6 applies to this situation and the Schmidt operator norms are a natural tool for tackling this problem. The following proposition is a natural starting point.

Proposition 5.7. *Let $\rho_\alpha \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n)$ be a Werner state. Then ρ^Γ is k-block positive if and only if $\alpha \leq \frac{1}{k}$.*

Proof. Simply note that $(n^2 - \alpha n)\rho_\alpha^\Gamma = I - \alpha E$ has only two distinct eigenvalues: 1 and $1 - \alpha n$. Corollary 5.6 then implies that ρ_α^Γ is k-block positive if and only if $\big\|E\big\|_{S(k)} \leq \frac{1}{\alpha n}$. We saw in Example 4.3 that $\big\|E\big\|_{S(k)} = \frac{k}{n}$, so the result follows. □

The special case $k = n$ of the above proposition is very well-known and states that ρ_α is PPT if and only if $\alpha \leq \frac{1}{n}$. Proposition 5.7 also shows that Werner states can not be bound-entangled for $\alpha > \frac{1}{2}$, which is also well-known. It has been conjectured [5, 26] that Werner states are bound-entangled for all $\alpha \leq \frac{1}{2}$; this is exactly the set of values for which ρ_α^Γ is 2-positive.

Although we now have determined k-block positivity of ρ_α^Γ, determining k-block positivity of $(\rho_\alpha^\Gamma)^{\otimes r}$ for $r > 1$ is not so simple in general because the projection onto the negative eigenspaces is no longer rank-1, so we cannot exactly compute its Schmidt k-norm. Additionally, $(\rho_\alpha^\Gamma)^{\otimes r}$ has more than two distinct eigenvalues in general so we can no longer use Corollary 5.6. To simplify the problem somewhat, consider the $\alpha = \frac{2}{n}$ case. Then the operator $X := (n^2 - 2)\rho_{2/n} = I - \frac{2}{n}E^\Gamma$ has eigenvalues 1 and -1, so $(X^\Gamma)^{\otimes r}$ has only two distinct eigenvalues (1 and -1) regardless of r. Corollary 5.6 then says that $\rho_{2/n}$ is bound entangled if and only if $\left\|P_r^-\right\|_{S(2)} \leq \frac{1}{2}$ for all $r \geq 1$, where P_r^- is the projection onto the -1 eigenspace of $(\rho_{2/n}^\Gamma)^{\otimes r}$. This mirrors the approach attempted in [4] to find a bound entangled NPPT Werner state, though that paper considers the $\alpha = \frac{1}{2}$ case instead. Note in particular that these tests of k-positivity are strong enough to determine bound entanglement in some cases, assuming we can compute or find strong bounds on these norms in this situation. Contrast this with the spectral tests of Chruściński and Kossakowski, which were shown to be unable to detect bound entanglement in [3].

We will finish this section by showing that, in the limit as r tends to infinity, it is not possible to do any better than $\left\|P_r^-\right\|_{S(2)} \leq \frac{1}{2}$. More precisely, it is the case that

$$\lim_{r\to\infty} \left\|P_r^-\right\|_{S(2)} \geq \frac{1}{2}.$$

To see this claim, observe that $\mathrm{rank}(P_1^-) = 1$ and $P_r^- = P_1^- \otimes P_{r-1}^+ + P_1^+ \otimes P_{r-1}^-$ for all $r \geq 2$, where P_r^+ is the projection onto the $+1$ eigenspace of $(\rho_{2/n}^\Gamma)^{\otimes r}$. It follows that $\mathrm{rank}(P_r^-) = \mathrm{rank}(P_{r-1}^+) + (n^2-1)\mathrm{rank}(P_{r-1}^-)$ for all $r \geq 2$. Standard techniques for solving recurrence relations then show that $\mathrm{rank}(P_r^-) = \frac{1}{2}(n^{2r} - (n^2-2)^r)$ for all $r \geq 1$. Plugging this into the lower bound of Theorem 4.14 reveals that

$$\left\|P_r^-\right\|_{S(2)} \geq \frac{n^{2r} + \frac{1}{2}(n^r-2)(n^{2r}-(n^2-2)^r)}{n^{2r}(n^r-1)} = \frac{n^r-2}{2(n^r-1)} - \frac{(n^r-2)(n^2-2)^r - 2n^{2r}}{2n^{2r}(n^r-1)}.$$

It is not difficult to verify that the lower bound on the right is always, for $n \geq 4$, strictly less than $\frac{1}{2}$. Furthermore, as $r \to \infty$, the rightmost fraction tends to zero and the left fraction tends to $\frac{1}{2}$. This shows that, asymptotically, $\frac{1}{2}$ is the smallest that we could ever hope $\left\|P_r^-\right\|_{S(2)}$ to be. It follows that $\rho_{2/n}$ is bound entangled if and only if

$$\lim_{r\to\infty} \left\|P_r^-\right\|_{S(2)} = \frac{1}{2}.$$

6. Other Connections in Quantum Information Theory

The Schmidt vector norms have a simple interpretation in quantum information theory, as $\left\||v\rangle\right\|_{s(k)}$ can be seen as a measure of how close the pure state $|v\rangle$ is to having Schmidt rank of k or less. Corollary 3.3 supports this interpretation, as we saw that $\left\||v\rangle\right\|_{s(k)} = 1$ (the largest possible value that norm can take on pure states) if and only if $SR(|v\rangle) \leq k$. On the other hand, consider the maximally-entangled state $|e\rangle := \frac{1}{\sqrt{n}}\sum_{i=1}^n |e_i\rangle \otimes |e_i\rangle \in \mathcal{H}_n \otimes \mathcal{H}_n$. Corollary 3.3 also implies that $\left\||e\rangle\right\|_{s(k)} = \frac{k}{n}$, which is the smallest that norm can ever be on pure states.

For a general mixed state ρ, one might want to think of $\left\|\rho\right\|_{S(k)}$ as measuring how close ρ is to having Schmidt number of k or less, but this interpretation is not quite right. Consider the following example, which shows that, in contrast to the Schmidt vector norm case, it is not the case that $SN(\rho) \leq k$ implies $\left\|\rho\right\|_{S(k)} = \left\|\rho\right\|$.

Example 6.1. Let $\rho \in \mathcal{L}(\mathcal{H}_2) \otimes \mathcal{L}(\mathcal{H}_2)$ have the following matrix representation in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$\rho = \frac{1}{5}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

It is clear that $SN(\rho) = 1$. However, the eigenvector corresponding to the (distinct) maximal eigenvalue 0.8606 is $|v\rangle := (0.6011, 0.4614, 0.4614, 0.4614)^T$. It is easily verified that $SR(|v\rangle) = 2$, so $\big\|\rho\big\|_{S(1)} < \big\|\rho\big\|$.

Nonetheless, it is the case that if the eigenspace corresponding to the maximal eigenvalue of ρ contains a state $|v\rangle$ with $SR(|v\rangle) \leq k$ then $\big\|\rho\big\|_{S(k)} = \big\|\rho\big\|$. More importantly though, we will see in the next section that the correct interpretation of $\big\|\rho\big\|_{S(k)}$ is as a measure of how close ρ is to a *pure* state $|v\rangle$ with $SR(|v\rangle) \leq k$.

6.1. **Quantum Fidelity and Trace Distance.** We will now link these norms to other distance measures that are commonly used in quantum information. Recall the *trace distance* δ and the *quantum fidelity* F between two density operators $\rho, \sigma \in \mathcal{L}(\mathcal{H})$:

$$\delta(\rho, \sigma) := \frac{1}{2}\mathrm{Tr}(|\rho - \sigma|),$$
$$F(\rho, \sigma) := \Big(\mathrm{Tr}\big(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\big)\Big)^2.$$

The trace distance can be thought of as the distance between ρ and σ, and the quantum fidelity can be interpreted as the amount of overlap between them. The following proposition reinforces again our interpretation of $\big\||v\rangle\big\|_{s(k)}$ as being a measure of how close $|v\rangle$ is to having Schmidt rank at most k, this time by characterizing both the maximal overlap that $|v\rangle$ has with states of small Schmidt rank, as well as the minimal distance that $|v\rangle$ is from such states. Similarly, $\big\|\rho\big\|_{S(k)}$ is seen to be a measure of how close ρ is to a pure state $|v\rangle$ with $SR(|v\rangle) \leq k$.

Proposition 6.2. *Let $\rho \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ and let $|w\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$. Then*

$$\big\|\rho\big\|_{S(k)} = \sup_{|v\rangle}\Big\{F(\rho, |v\rangle\langle v|) : SR(|v\rangle) \leq k\Big\} \text{ and} \tag{6}$$

$$\sqrt{1 - \big\||w\rangle\big\|_{s(k)}} = \inf_{|v\rangle}\Big\{\delta(|w\rangle\langle w|, |v\rangle\langle v|) : SR(|v\rangle) \leq k\Big\}. \tag{7}$$

Furthermore, if ρ is a pure state then

$$\big\|\rho\big\|_{S(k)} = \sup_{\sigma}\Big\{F(\rho, \sigma) : SN(\sigma) \leq k\Big\}. \tag{8}$$

Proof. It is well-known that for pure states $|v\rangle$, $F(|v\rangle\langle v|, \sigma) = \langle v|\sigma|v\rangle$. It follows immediately that

$$\big\|\rho\big\|_{S(k)} = \sup_{|v\rangle}\Big\{\langle v|\rho|v\rangle : SR(|v\rangle) \leq k\Big\} = \sup_{|v\rangle}\Big\{F(\rho, |v\rangle\langle v|) : SR(|v\rangle) \leq k\Big\}.$$

To see Equation (8) if $\rho = |w\rangle\langle w|$, observe that Corollary 4.5 implies

$$\big\||w\rangle\langle w|\big\|_{S(k)} = \sup_{\sigma}\Big\{\mathrm{Tr}(|w\rangle\langle w|\sigma) : SN(\sigma) \leq k\Big\} = \sup_{|v\rangle}\Big\{F(|w\rangle\langle w|, \sigma) : SN(\sigma) \leq k\Big\}.$$

Equation (7) holds simply by a standard relationship between the trace distance and quantum fidelity when both of the inputs are pure states. □

6.2. **Other Applications of Norms of Projections.** We have seen that the Schmidt norms of orthogonal projections have several applications within quantum information theory. One more reason for studying these norms comes from their appearance in a conjecture of Brandao [11], which asks whether or not there exists a $0 < \varepsilon < 1$ such that, for all n and all orthogonal projections $P = P^* = P^2 \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n)$,

$$\big\|P\big\|_{S(1)} \geq \sqrt{\frac{\mathrm{rank}(P)}{n^{2+\varepsilon}}}. \tag{9}$$

Recall that Corollary 4.13 and Theorem 4.14 say that

$$\big\|P\big\|_{S(1)} \geq \frac{1}{\big\lceil\big(n - \sqrt{\mathrm{rank}(P) - 1}\big)\big\rceil} \quad \text{and} \tag{10}$$

$$\big\|P\big\|_{S(1)} \geq \frac{\mathrm{rank}(P)}{n^2}. \tag{11}$$

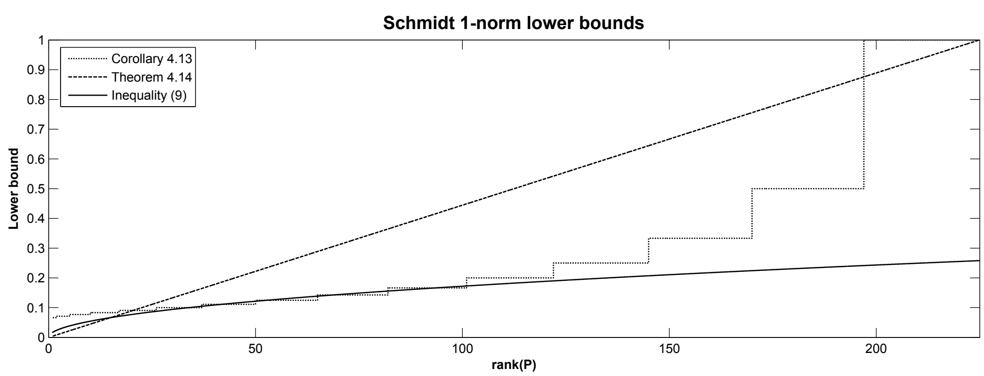

FIGURE 1. A comparison of the various lower bounds for the Schmidt norm on projections in the $n = m = 15$ and $k = 1$ case. Corollary 4.13 provides the best bound when rank(P) is very low or high, but Theorem 4.14 provides the best bound when rank(P) is moderate. Taken together, these bounds show that Inequality (9) holds for all fixed, finite n (Inequality (9) is shown with $\varepsilon = 0.99$).

For any fixed $n \in \mathbb{N}$ and rank$(P) \leq n$, Inequality (10) implies (9) for some $\varepsilon_n < 1$. Similarly, if rank$(P) > n$ then Inequality (11) implies (9) for some $\varepsilon_n < 1$. Thus, the conjecture of Brandao holds in any *fixed* finite dimension.

Nonetheless, Inequality (9) can be seen not to hold as n tends to infinity when $\mathrm{rank}(P) = n$ via methods of convex geometry [27]. That is, even though there exists an $\varepsilon_n < 1$ for each n such that Inequality (9) holds, it is the case that $\lim_{n\to\infty} \varepsilon_n = 1$, so there is no global ε satisfying Inequality (9).

In particular, there exists a universal constant $c \in \mathbb{R}$ such that for any n there exists a projection P of rank n such that $\big\|P\big\|_{S(1)} \leq \frac{c}{n}$. However, the conjecture remains open as Inequality (9) only needs to hold for certain projections to have important implications. If it were true, it would imply that the regularized relative entropy of entanglement [7, 8] is super-additive [28]. This in turn would imply that $QMA(k) = QMA(2)$ for all $k > 2$ via a result of Aaronson et. al. [29].

7. Bounding Schmidt Operator Norms via Semidefinite Programs

We have seen in Proposition 4.2 that we can compute the Schmidt operator norms of rank-1 operators. However, Corollary 4.7 showed that that computing these norms in full generality is an extremely difficult problem. In this section we develop a family of semidefinite programs that can be used to provide upper bounds on the Schmidt operator norms and compute them exactly in low-dimensional cases. However, our primary motivation for looking at semidefinite programs comes from their duality theory, which leads to some simple theoretical results that further establish the link between the Schmidt operator norms and k-block positive operators.

7.1. **Introduction to Semidefinite Programs.** Here we introduce the reader to semidefinite programming (SP), which we will see provides a step in the direction of being able to compute the Schmidt operator norms. Our introduction will be brief – for a more in-depth introduction and discussion, the reader is encouraged to read any of a number of other sources including [30, 31, 32, 33, 34]. Importantly, there are explicit methods that are able to approximately solve semidefinite programs of the type presented in this paper to any desired accuracy in polynomial time [35].

For our purposes, assume we have a Hermicity-preserving linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$, two operators $A \in \mathcal{L}(\mathcal{H}_n)$ and $B \in \mathcal{L}(\mathcal{H}_m)$, and a convex cone $\mathcal{C} \subseteq (\mathcal{L}(\mathcal{H}_n))^+$. Then the corresponding semidefinite program is given by the following pair of optimization problems:

$$
\begin{array}{rll@{\qquad}rl}
& \multicolumn{2}{c}{\textbf{Primal problem}} & \multicolumn{2}{c}{\textbf{Dual problem}} \\
& \text{maximize:} & \mathrm{Tr}(AX) & \text{minimize:} & \mathrm{Tr}(BY) \\
(12) & \text{subject to:} & \Phi(X) \leq B & \text{subject to:} & \Phi^\dagger(Y) \geq A \\
& & X \in \mathcal{C} & & Y \in \mathcal{C}^O
\end{array}
$$

The form (12) differs slightly from the standard form of semidefinite programs, but it is equivalent and more well-suited to our particular needs. A similar form has been used very recently to solve other problems in quantum information [36, 37]. We define the *primal feasible* set $\mathcal{A}$ and *dual feasible* set $\mathcal{B}$ to be

$$\mathcal{A} := \big\{X \in \mathcal{C} : \Phi(X) \leq B\big\} \qquad \text{and} \qquad \mathcal{B} := \big\{Y \in \mathcal{C}^O : \Phi^\dagger(Y) \geq A\big\}.$$

The optimal values associated with the primal and dual problems are defined to be

$$\alpha := \sup_{X\in\mathcal{A}} \big\{\mathrm{Tr}(AX)\big\} \qquad \text{and} \qquad \beta := \inf_{Y\in\mathcal{B}} \big\{\mathrm{Tr}(BX)\big\},$$

and if $\mathcal{A}$ or $\mathcal{B}$ is empty then we set $\alpha = -\infty$ or $\beta = \infty$, respectively.

Semidefinite programming has a strong theory of duality. The theory of weak duality tells us it is always the case that $\alpha \leq \beta$. Equality is actually attained for many semidefinite programs of interest though, as the following theorem shows.

Theorem 7.1 (Strong duality)**.** *The following two implications hold for every semidefinite program of the form* (12)*.*

1. *Strict primal feasibility: If β is finite and there exists an operator X in the interior of $\mathcal{C}$ such that $\Phi(X) < B$, then $\alpha = \beta$ and there exists $Y \in \mathcal{B}$ such that* $\mathrm{Tr}(YB) = \beta$.
2. *Strict dual feasibility: If α is finite and there exists an operator Y in the interior of $\mathcal{C}^O$ such that $\Phi^\dagger(Y) > A$, then $\alpha = \beta$ and there exists $X \in \mathcal{A}$ such that* $\mathrm{Tr}(XA) = \alpha$.

There are other conditions that imply strong duality, but the conditions of Theorem 7.1 (which are known as *Slater-type conditions*) will be sufficient for our needs.

7.2. **A Family of Semidefinite Programs for Schmidt Operator Norms.** Given a positive operator $X \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$ and a natural number k, we will now present a family of semidefinite programs with the following properties:

- Strong duality (i.e., $\alpha = \beta$) holds for each semidefinite program.
- The optimal value α of each SP is an upper bound of $\big\|X\big\|_{S(k)}$.
- There is an SP in the family such that the optimal value satisfies $\alpha = \big\|X\big\|_{S(k)}$.

Let $X \in (\mathcal{L}(\mathcal{H}_n)\otimes\mathcal{L}(\mathcal{H}_m))^+$ be a positive operator for which we wish to compute $\big\|X\big\|_{S(k)}$. Let $\Phi_k : \mathcal{L}(\mathcal{H}_m) \to \mathcal{L}(\mathcal{H}_m)$ be a fixed k-positive linear map and consider the following semidefinite program:

(13)

Primal problem		**Dual problem**	
maximize:	$\mathrm{Tr}(X\rho)$	minimize:	λ
subject to:	$(id_n \otimes \Phi_k)(\rho) \geq 0$	subject to:	$\lambda I_n \otimes I_m \geq (id_n \otimes \Phi_k^\dagger)(Y) + X$
	$\mathrm{Tr}(\rho) = 1$		$Y \geq 0$
	$\rho \geq 0$		

This semidefinite program isn't quite in the form of (12), so we first check that these problems are indeed duals of each other and form a valid semidefinite program. To this end, consider the linear map $\Psi : \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m) \to (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)) \oplus \mathcal{L}(\mathcal{H}_1)$ defined by

$$\Psi(\rho) = \begin{bmatrix} -(id_n \otimes \Phi_k)(\rho) & 0 \\ 0 & \mathrm{Tr}(\rho) \end{bmatrix}.$$

Then the dual map $\Psi^\dagger : (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)) \oplus \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is given by

$$\Psi^\dagger\Big(\begin{bmatrix} Y & v \\ w^* & \lambda \end{bmatrix} \Big) = \lambda I_n \otimes I_m - (id_n \otimes \Phi_k^\dagger)(Y).$$

Finally, setting

$$A := X \quad \text{and} \quad B := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and recalling that the convex cone of positive semidefinite operators is its own dual cone gives the semidefinite program (13) in standard form.

We now show that this program satisfies the Slater-type conditions for strong duality given by Theorem 7.1. It is clear that both α and β are finite, as $\mathrm{Tr}(X\rho) \leq \big\|X\big\|$ and $\lambda \geq 0$. Both feasible sets are also non-empty (for example, one could take ρ to be any separable state, $Y = 0$, and $\lambda \geq \big\|X\big\|$). Strong dual feasibility then follows by choosing any $Y > 0$ and a sufficiently large λ. Strong primal feasibility is not necessarily satisfied, however, as there is no guarantee that Φ_k does not introduce singularities in ρ (for example, consider the zero map, which is k-positive). We could restrict the family of k-positive maps that we are interested in if we really desired strong primal feasibility, but strict dual feasibility is enough for our purposes.

It follows from Theorem 2.2 and Corollary 4.5 that, for any k-positive map Φ_k, the optimal value of the semidefinite program (13) is an upper bound of $\big\|X\big\|_{S(k)}$ – the supremum in the primal problem is just being taken over a set that is larger than the set of operators ρ with $SN(\rho) \leq k$. This leads to the following theorem.

Theorem 7.2. *Let $X \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$. Then*

$$\big\|X\big\|_{S(k)} = \inf_Y \big\{ \big\|X + Y\big\| : Y \text{ is } k\text{-block positive} \big\}.$$

Proof. Because Φ_k is k-positive if and only if $\Phi_k^\dagger$ is k-positive, the dual problem (13) can be rephrased as asking for the infimum of $\big\|X + Y\big\|$, where the infimum is taken over a subset of the k-block positive operators $Y \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$. The preceding paragraph showed us that

$$\big\|X\big\|_{S(k)} \leq \inf_Y \big\{ \big\|X + Y\big\| : Y \text{ is } k\text{-block positive} \big\}.$$

To see that equality is attained, choose $Y := \big\|X\big\|_{S(k)} I - X$, which we know from Corollary 4.7 is k-block positive. Then

$$\big\|X + Y\big\| = \big\|X + \big\|X\big\|_{S(k)} I - X\big\| = \big\|X\big\|_{S(k)}.$$

□

In fact, it is not difficult to see that there is a particular k-positive map Φ_k such that $\big\|X\big\|_{S(k)}$ is attained as the optimal value of the semidefinite program (13) corresponding to Φ_k – simply let Φ_k be the map associated with the operator $\big\|X\big\|_{S(k)} I - X$ via the Choi-Jamiolkowski isomorphism.

One additional obvious implication of Theorem 7.2 is that $\big\|X\big\|_{S(k)} \leq \big\|X + Y\big\|$ for all $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ and all k-block positive $Y \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$. The following corollary shows that this can be strengthened into another characterization of k-positivity.

Corollary 7.3. *Let $Y = Y^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$. Then Y is k-block positive if and only if*

$$\big\|X\big\|_{S(k)} \leq \big\|X + Y\big\| \quad \forall\, X \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+.$$

Proof. The "only if" direction of the proof follows immediately from Theorem 7.2. To see the "if" direction, assume that Y is not k-block positive and choose $X = cI - Y$, where $c \in \mathbb{R}$ is large enough that $cI - Y \geq 0$. Then, because Y is not k-block positive, there exists a vector $|v\rangle$ with $SR(|v\rangle) \leq k$ such that $\langle v|Y|v\rangle < 0$. Thus

$$\big\|X\big\|_{S(k)} \geq \langle v|(cI - Y)|v\rangle = c - \langle v|Y|v\rangle > c = \big\|X + Y\big\|.$$

□

Recall that if $m = 2$ and $n = 2$ or $n = 3$ then the transpose map T alone is enough to determine whether or not ρ is separable (i.e., $SN(\rho) = 1$ if and only if $(id_n \otimes T)(\rho) \geq 0$). It follows that the semidefinite program (13) with $k = 1$ and $\Phi_1 = T$ can be used to compute $\big\|X\big\|_{S(1)}$ for positive operators X. That is, the infinite family of semidefinite programs reduces to just a single semidefinite program in this situation. We can then use Corollary 4.7 to determine 1-block positivity of operators $X \in \mathcal{L}(\mathcal{H}_3) \otimes \mathcal{L}(\mathcal{H}_2)$.

8. Norms Restricted to Other Convex Cones of Operators

We will now see that many of the results for the Schmidt operator k-norms of operators actually hold in the much more general setting of arbitrary convex mapping cones of operators. We will begin by defining the notion of a mapping cone, which originally appeared in [9].

Definition 8.1. *Let $\mathcal{S} \subseteq \mathcal{L}(\mathcal{L}(\mathcal{H}_n), \mathcal{L}(\mathcal{H}_m))$ be a cone of completely positive linear maps. $\mathcal{S}$ is said to be a* mapping cone *if $\Phi \circ \Psi \in \mathcal{S}$ and $\Psi \circ \Phi \in \mathcal{S}$ whenever $\Phi \in \mathcal{S}$ and Ψ is completely positive.*

Mapping cones appeared recently in [15] as a way of generalizing the dual cone relationships between k-block positive operators and operators with Schmidt number no greater than k. These dual relationships can be seen implicitly in the semidefinite programming results of the last section, so it is no surprise that the notion of mapping cones provides a natural generalization in this setting as well. Mapping cones can be defined without the restriction that they be a subset of the completely positive maps, though the definition provided will be better-suited to our purposes.

We will say that a cone of operators $\mathcal{C} \subseteq (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$ is a mapping cone if the cone of associated linear maps (via the Choi-Jamiolkowski isomorphism) is a mapping cone. If necessary, we will specify whether we mean a mapping cone of operators or a mapping cone of linear maps, but our meaning should be clear from context.

Definition 8.2. *Let $X \in \mathcal{L}(\mathcal{H})$ and let $\mathcal{C} \subseteq (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$ be a convex cone. Then we define the $\mathcal{C}$-*operator norm *of X, denoted $\big\|X\big\|_{\mathcal{C}}$, by*

$$\big\|X\big\|_{\mathcal{C}} := \sup_{\rho \in \mathcal{C}} \Big\{ \big|\mathrm{Tr}(X\rho)\big| \Big\}.$$

It is easy to see that this defines a valid norm if $\mathcal{C}$ is a mapping cone. It is also a norm for many other convex cones of interest – all that needs to be checked is that $\mathcal{C}$ is large enough that $\mathrm{Tr}(X\rho) = 0 \ \ \forall \rho \in \mathcal{C}$ implies $X = 0$.

Observe that if $\mathcal{C} = \mathcal{S}_k$ then this definition reduces to exactly the Schmidt k-norm of X when X is normal – the defining equation is easily seen to be in analogy with Corollary 4.5.

The norm $\|X\|_{\mathcal{C}}$ has a similar interpretation to that of the Schmidt operator norms as well. We can think of $\|X\|_{\mathcal{C}}$ as roughly measuring how close X is to an operator in $\mathcal{C}$. Throughout the rest of this section, we will assume that $\mathcal{C}$ is a convex mapping cone.

It is trivial to see that if $\mathcal{C} \subseteq \mathcal{D}$, where $\mathcal{D}$ is another convex mapping cone, then $\|X\|_{\mathcal{C}} \leq \|X\|_{\mathcal{D}}$. In particular this implies that $\|X\|_{\mathcal{C}} \leq \|X\|$ always. Additionally, several of the characterizations of Schmidt operator norms carry over in an obvious way to this more general setting.

Proposition 8.3. *Let $X \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$ be positive. Then $cI - X \in \mathcal{C}^O$ if and only if $c \geq \|X\|_{\mathcal{C}}$.*

Proof. By definition, $cI - X \in \mathcal{C}^O$ if and only if

$$\mathrm{Tr}((cI - X)\rho) = c - \mathrm{Tr}(X\rho) \geq 0 \quad \forall\, \rho \in \mathcal{C}.$$

This if true if and only if $c \geq \|X\|_{\mathcal{C}}$, completing the proof. □

Now let $X \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$ be positive and consider the following semidefinite program.

$$\begin{array}{llll} & \textbf{Primal problem} & & \textbf{Dual problem} \\ (14) & \text{maximize:} \ \ \mathrm{Tr}(X\rho) & & \text{minimize:} \ \ \lambda \\ & \text{subject to:} \ \ \mathrm{Tr}(\rho) = 1 & & \text{subject to:} \ \ \lambda I_n \geq Y + X \\ & \qquad\qquad\ \ \rho \in \mathcal{C} & & \qquad\qquad\ \ Y \in \mathcal{C}^O \end{array}$$

It is easy to see that these problems are indeed duals of each other and form a valid semidefinite program, using the exact same method as was used in Section 7.2 to show that the semidefinite program (13) is valid. Strong dual duality also holds in this setting. The main difference here is that we have $\rho \in \mathcal{C}$ and $Y \in \mathcal{C}^O$ rather than $\rho, Y \geq 0$ – we could have stated the semidefinite program (13) in terms of the cone $\mathcal{S}_k$, but then it would become less clear how to compute upper bounds of $\|X\|_{S(k)}$ using k-positive maps.

Just as in the case for the Schmidt operator norms, the theory of semidefinite programming leads to the following two results. We state them without proof, as their proofs are almost identical to the proofs of Theorem 7.2 and Corollary 7.3, respectively.

Theorem 8.4. *Let $X \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+$ be positive. Then*

$$\|X\|_{\mathcal{C}} = \inf_{Y} \big\{ \|X + Y\| : Y \in \mathcal{C}^O \big\}.$$

Corollary 8.5. *Let $Y = Y^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$. Then $Y \in \mathcal{C}^O$ if and only if*

$$\|X\|_{\mathcal{C}} \leq \|X + Y\| \quad \forall\, X \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+.$$

8.1. **Application to PPT States.** Given any k-positive linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \to \mathcal{L}(\mathcal{H}_m)$, there exists a natural convex cone $\mathcal{C}_\Phi$ associated with Φ:

$$\mathcal{C}_\Phi := \Big\{ X \in (\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m))^+ : (id_n \otimes \Phi)(X) \geq 0 \Big\}.$$

Given any such convex cone, $\|\cdot\|_{\mathcal{C}_\Phi}$ is indeed a norm and we are able to compute $\|X\|_{\mathcal{C}_\Phi}$ for $X \geq 0$ to any desired accuracy via semidefinite programming, as seen in the previous section. In fact, $\big\|X\big\|_{\mathcal{C}_{\Phi_k}}$ is exactly what is computed by the semidefinite program (13). It follows that $\big\|X\big\|_{S(k)} = \inf_{\Phi_k}\{\big\|X\big\|_{\mathcal{C}_{\Phi_k}} : \Phi_k \text{ is } k-\text{positive}\}$.

In the case of the transpose map $T : \mathcal{L}(\mathcal{H}_n) \to \mathcal{L}(\mathcal{H}_n)$, $\mathcal{C}_T$ is exactly the cone of unnormalized PPT states, and so the norm $\big\|\cdot\big\|_{\mathcal{C}_T}$ can be seen as a measure of how close a given operator is to having positive partial transpose. It is known [10] that the dual cone of the PPT states is given by

$$\mathcal{C}_T^O = \Big\{X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m) : X = Y + Z \text{ for some } Y \geq 0, (id_n \otimes T)(Z) \geq 0\Big\}.$$

This leads immediately to the following characterizations of $\big\|\rho\big\|_{\mathcal{C}_T}$ via Theorem 8.4.

Proposition 8.6. *Let $\rho \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ be a density operator. Then*

$$\big\|\rho\big\|_{\mathcal{C}_T} = \inf_Y \big\{\big\|\rho + Y\big\| : (id_n \otimes T)(Y) \geq 0\big\}.$$

9. Outlook

We have seen that Schmidt norms play an important role in quantum information theory and have actually been used implicitly several times over the past decade. They are powerful tools for determining k-positivity of Hermitian operators, especially for the partial transpose of Werner states and other operators with only two distinct eigenvalues. Further exploration of the relationship between these norms and Werner states in search of NPPT bound entangled states is warranted.

Many of the applications of Schmidt norms involve only the value of the norm on orthogonal projections. While we have seen several ways to bound these norms, we have no reason to believe that our best lower bound involving n, k, and the rank of the projection is tight. A tight lower bound would be of significant interest, as would a characterization of the projections that attain the lower bound.

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