LEMMA OF THE MONTH #3
SYLVESTER'S LAW AND $*$-CONGRUENCE

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Square matrices A, B are said to be $*$-congruent if there is an invertible matrix S such that $SAS^* = B$. Similar to the Jordan canonical form of matrices as the canonical form under similarity, we would like to have a canonical form for matrices under $*$-congruence. The most well-known result along these lines is the following theorem of Sylvester.

Theorem 1 (Sylvester's Law of Inertia [1]). *Let $A, B \in M_n$ be Hermitian. Then A and B are $*$-congruent if and only if they have the same number of positive, zero, and negative eigenvalues.*

This leads to the following simple canonical form for Hermitian matrices under $*$-congruence:

$$SAS^* = \begin{bmatrix} I_j & 0 & 0 \\ 0 & -I_k & 0 \\ 0 & 0 & 0_m \end{bmatrix}, \tag{1}$$

where j is the number of positive eigenvalues of A, k is the number of negative eigenvalues of A, and m is the number of zero eigenvalues of A.

There is a very straightforward generalization of this result to normal matrices due to Ikramov [2], which we prove here.

Theorem 2. *Let $A, B \in M_n$ be normal. Then A and B are $*$-congruent if and only if they have the same number of eigenvalues on each open ray from the origin in the complex plane.*

Proof. By the Spectral Theorem, we can find a unitary U and a diagonal matrix D_1 such that $D_1 = UAU^*$. Define $D_2 := \sqrt{D^*D}^{-1/2}$ and observe that $D_2UAU^*D_2^*$ is diagonal and each of its entries has modulus zero or one. The "if" direction of the proof follows immediately by applying the same procedure to B and using transitivity of $*$-congruence.

To see the "only if" direction, we may assume that A is diagonal by the Spectral Theorem. Using $A = SBS^*$ and the Spectral Theorem on B then gives a unitary U and a diagonal matrix D_1 such that $A = S(U^*D_1U)S^*$. It follows via a simple entry-wise calculation that this implies $D_2 := SU^*$ is diagonal, and $A = D_2D_1D_2^*$. Since the D_1 has the same eigenvalues as B and the conjugation by D_2 only scales the eigenvalues, the proof is complete. $\square$

Note that Ikramov's Theorem naturally implies a canonical form for normal matrices analogous to that of Equation (1), but with the 1's and -1's on the diagonal replaced by arbitrary complex numbers of modulus 1.

Finally, there *is* a $*$-congruence canonical form for general matrices, but it is quite technical. The interested reader can find it in [3].

References

[1] Sylvester, J. J., *A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares.* Philosophical Magazine **IV**: 138142 (1852).

[2] Ikramov, Kh. D., *On the inertia law for normal matrices.* Doklady Math. **64** (2001) 141-142.

[3] Horn, R. and Sergeichuk, V., *Canonical forms for complex matrix congruence and *congruence.* Linear Algebra Appl. **416** (2006) 1010-1032. arXiv:0709.2473v1 [math.RT]

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