PARTIALLY ENTANGLEMENT BREAKING MAPS AND RIGHT CP-INVARIANT CONES

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Abstract. There are many results in quantum computing that give ways of characterizing classes of density matrices in terms of classes of linear maps, and vice-versa – Horodecki's theorem for positive entanglement witnesses, Terhal and Horodecki's theorem for density matrices with Schmidt Number at most k, and so on. The goal of this paper is to determine for which sets of linear maps and density matrices these types of relationships hold. Along the way a characterization of partially entanglement breaking maps is given, and it is shown that many of their well-known characterizations follow simply because they form a cone that is invariant under right composition with CP maps – a fact that is trivial to check.

1. Introduction

In the theory of quantum maps, many theorems exist to tie together entanglement of density operators and properties of linear maps. Probably the most well-known such theorem is one of the Horodeckis [5], which provides necessary and sufficient conditions for a state to be separable. In particular, the theorem says that a state ρ is separable if and only if $(id \otimes \Phi)(\rho)$ is positive for all positive linear maps Φ.

Separable states can also be characterized in terms of another class of linear maps: the entanglement breaking maps [7]. In particular, it is well-known that ρ is separable if and only if it is the Choi matrix [2] of an entanglement breaking map. Furthermore, these two characterizations of separable states seem to be closely related, as the cone of positive linear maps is exactly the dual cone of the entanglement breaking maps.

In this paper, generalizations of these results will be explored to answer the question of *why* these inter-relationships exist between these particular families of maps and states. In Section 2 some required definitions and preliminaries will be presented – the Schmidt Rank and Schmidt Number of states, as well as some simple generalizations of well-known results. Section 3 will introduce the reader to the partially entanglement breaking maps, and some results will be presented that generalize the aforementioned relationships between separable states, positive maps and entanglement breaking maps.

Section 4 will introduce the reader to cones of linear maps and dual cones. In Section 4.3 it will be shown that the dual cone of the k-positive maps is exactly the set of k-PEB maps (and vice-versa), and in Section 4.4 more general cones of positive linear maps will be explored, and results from earlier in the paper will be proved (and generalized) using only the dual cone relationship of the maps.

2. Preliminaries

2.1. **Notation.** The following notational conveniences will be used frequently in what follows. They will be re-introduced at relevant points throughout the paper, but they are provided here as a simple reference for the reader.

- The vector $|e\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is defined as $|e\rangle := \sum_{j=1}^n |e_j\rangle \otimes |e_j\rangle$,
- $\mathcal{P}_k$ is the set of k-positive linear maps from M_m to M_n,
- $\mathcal{CP}$ is the set of completely positive linear maps,
- $\mathcal{B}_k$ is the set of k-PEB maps (to be defined in Section 3),
- C_Φ denotes the Choi matrix of the linear map Φ, and
- $C_{\mathcal{CP}}$ denotes the set of Choi matrices of CP maps ($C_{\mathcal{P}_k}$ and $C_{\mathcal{B}_k}$ are defined analogously).

Also note that for this paper the convention will be made that the inner product $\langle\phi|\psi\rangle$ will be linear in the *second* variable and conjugate linear in the *first* variable; this convention is made both to retain consistency with the relevant literature on partially entanglement breaking maps, and to simplify many inner product and outer product tricks used in proofs.

2.2. **Schmidt Decomposition Theorem.** The Schimdt Decomposition Theorem provides a first step toward defining the Schmidt Number of a density operator, which is one of our first main goals. As will be seen in its proof, the Schmidt Decomposition Theorem is essentially the Singular Value Decomposition Theorem in disguise.

Theorem 2.1 (Schmidt Decomposition)**.** *For any $|\phi\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ there exist orthonormal sets of vectors $\{|u_1\rangle, |u_2\rangle, \ldots, |u_m\rangle\} \subset \mathbb{C}^m$ and $\{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\} \subset \mathbb{C}^n$ such that*

$$|\phi\rangle = \sum_{j=1}^{\min\{m,n\}} \alpha_j |u_j\rangle \otimes |v_j\rangle \tag{1}$$

for some non-negative real constants $\{\alpha_j\}$.

Proof. Assume that $n \leq m$, as it will be clear how to modify the proof if the opposite inequality holds. Note that for any $|\psi_1\rangle \in \mathbb{C}^m$ and $|\psi_2\rangle \in \mathbb{C}^n$, we can isomorphically associate $|\psi_1\rangle \otimes \overline{|\psi_2\rangle} \in \mathbb{C}^m \otimes \mathbb{C}^n$ with $|\psi_1\rangle\langle\psi_2| \in M_{m,n}$ (in fact, this isomorphism is isometric if the norm we put on $M_{m,n}$ is the Frobenius norm). Since this is an isomorphism, it is linear and so we can associate any vector $|\phi\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ with a matrix $A_\phi \in M_{m,n}$. The Singular Value Decomposition tells us then that there exist unitaries $U \in M_m$, $V \in M_n$, and a positive semidefinite diagonal matrix $D \in M_m$ such that

$$A_\phi = U \begin{bmatrix} D \\ 0 \end{bmatrix} V.$$

If we expand out the previous matrix multiplication then we see that

$$A_\phi = \sum_{j=1}^n \alpha_j |u_j\rangle\langle v_j|,$$

where α_j is the j^{th} diagonal entry of D, $|u_j\rangle$ is the j^{th} column of U, and $\langle v_j|$ is the j^{th} row of V. Since U and V are both unitaries, simply associating A back with the vector

$$\sum_{j=1}^{n} \alpha_j |u_j\rangle \otimes \overline{|v_j\rangle} \in \mathbb{C}^m \otimes \mathbb{C}^n$$

completes the proof. □

2.3. **Schmidt Rank.** In the Schmidt Decomposition (1) of a vector $|\phi\rangle$, the least number of terms required in the summation is known as the *Schmidt Rank* of $|\phi\rangle$. By following along through the proof, we see that the Schmidt Rank of $|\phi\rangle$ is equal to the number of non-zero singular values of the matrix $A_\phi \in M_{m,n}$ to which $|\phi\rangle$ is associated. Thus, the Schmidt Rank of $|\phi\rangle$ is also equal to the rank of A_ϕ.

Further, since each $|u_j\rangle\langle v_j| \in M_{m,n}$ has rank 1, we see that even if we remove the requirement that the sets $\{|u_1\rangle, \ldots, |u_m\rangle\}$ and $\{|v_1\rangle, \ldots, |v_n\rangle\}$ be orthonormal, it is impossible to write $|\phi\rangle$ as the sum of fewer elementary tensors. This fact will be useful in proofs, because when trying to bound the Schmidt Rank of a vector, checking that these sets of vectors are orthonormal will not be needed.

The Schmidt Rank can roughly be interpreted as the "amount of entanglement" contained within a pure state. The following special cases give some motivation to interpretation:

- Separable pure states are represented exactly by the vectors with Schmidt Rank equal to 1.
- If $m = n$ then maximally-entangled pure states have Schmidt Rank equal to n (although the converse does not hold).

Note that the Schmidt Rank is only defined on pure states (ie., states that can be represented by single vectors). The *Schmidt Number* is the natural extension of the Schmidt Rank to arbitrary density matrices (ie., pure as well as mixed states).

2.4. **Schmidt Number.** Recall that density matrices can be written in the following form:

$$\rho = \sum_{j=1} |v_j\rangle\langle v_j| \in M_m \otimes M_n. \tag{2}$$

Definition 2.2 (Schmidt Number [14]). *The* Schmidt Number *of a density matrix ρ (denoted by $SN(\rho)$) is the least natural number k such that it can be written in the form* (2) *using vectors $|v_j\rangle$ with Schmidt Rank not greater than k.*

It is easy to see that for a pure state $|\phi\rangle$, the Schmidt Rank of $|\phi\rangle$ coincides with the Schmidt Number of $|\phi\rangle\langle\phi|$, as would be hoped. Also, the Schmidt Number of a density operator still tells us roughly "how entangled" that state is, as the following special cases suggest:

- Separable states are represented by density operators of the form $\rho = \sum_j \sigma_j \otimes \tau_j$, where each $\sigma_j, \tau_j \geq 0$. These are exactly the density operators with Schmidt Number equal to 1.

- If $m = n$ then maximally-entangled states have Schmidt Number equal to n (although the converse does not hold).

Furthermore, we have the very desirable (and intuitive) property that applying a quantum operation to one "piece" of a given state ρ can not increase its Schmidt Number. This is the content of the following theorem.

Theorem 2.3. *Let $\rho \in M_m \otimes M_n$ be a density matrix and let $\Phi : M_n \mapsto M_k$ be a completely positive map. Then*

$$SN((id_m \otimes \Phi)(\rho)) \leq SN(\rho).$$

Proof. Recall that a result of Choi [2] says that Φ can be written as $\Phi(X) = \sum_l A_l X A_l^\dagger$ for some family $\{A_l\} \in M_{k,n}$. Assume that $\rho = |\psi\rangle\langle\psi|$ is a pure state with $SN(\rho) = k$ and write $|\psi\rangle = \sum_{i=1}^k |v_i\rangle \otimes |w_i\rangle$ so that

$$\begin{aligned}(id_m \otimes \Phi)(\rho) &= \sum_{i,j=1}^k |v_i\rangle\langle v_j| \otimes \Phi(|w_i\rangle\langle w_j|) \\ &= \sum_{i,j=1}^k |v_i\rangle\langle v_j| \otimes \sum_l A_l |w_i\rangle\langle w_j| A_l^\dagger \\ &= \sum_l \Big(\sum_{i=1}^k |v_i\rangle \otimes A_l|w_i\rangle \Big) \Big(\sum_{i=1}^k |v_i\rangle \otimes A_l |w_i\rangle \Big)^\dagger,\end{aligned}$$

which is a density matrix with Schmidt Number no more than k. The result follows simply by linearity and the fact that pure states span all density matrices. □

In slightly more generality, it is well-known that the Schmidt Number of a state is non-increasing under local quantum operations and classical communication [14].

2.5. **Generalized Separability Criteria.** We saw earlier that determining the Schmidt Rank of a pure state isn't a particularly challenging task. The situation is much different for determining the Schmidt Number of an arbitrary state, however – determining the Schmidt Number of an arbitrary state (and even just whether the state is separable or not) is an NP-hard problem.[4]

Despite the hardness of finding the exact Schmidt Number of a given state ρ, we will soon see how we can bound $SN(\rho)$ from below by examining the action of maps of the form $(id_n \otimes \Phi)$ on ρ. It will be illustrative to start with a simple example.

Let $k \in \mathbb{N}$ and consider the map $\Phi_k : M_n \mapsto M_n$ defined by $\Phi_k(\rho) = k\mathrm{Tr}(\rho) \cdot I_n - \rho$. We will now see that if $\rho \in M_n \otimes M_n$ is such that $SN(\rho) \leq k$ then $(id_n \otimes \Phi)(\rho) \geq 0$, even though it is easy to check that Φ_k is not completely positive if $k < n$.

Proof that $SN(\rho) \leq k \Rightarrow (id_n \otimes \Phi_k)(\rho) \geq 0$. First notice that, due to linearity, it is enough to require that $(id_n \otimes \Phi)$ be positive on pure states with Schmidt Rank at most k. Thus, consider an arbitrary such pure state:

$$\rho = |\psi\rangle\langle\psi| = \sum_{i,j=1}^{k} |v_i\rangle\langle v_j| \otimes |w_i\rangle\langle w_j|,$$

where $\{|v_j\rangle\}, \{|w_j\rangle\} \subseteq \mathbb{C}^n$ are orthogonal sets (and the $|w_j\rangle$'s are in fact orthonormal).

First notice that $I_n \geq |w_i\rangle\langle w_i|$ implies that $kI_n - |w_i\rangle\langle w_i| \geq (k-1)|w_i\rangle\langle w_i|$. Because the $|w_j\rangle$'s are orthonormal it follows that

$$\begin{aligned}
(id_n \otimes \Phi_k)(\rho) &= \sum_{i,j=1}^{k} |v_i\rangle\langle v_j| \otimes (k\langle w_j|w_i\rangle I_n - |w_i\rangle\langle w_j|) \\
&\geq \sum_{i=1}^{k}(k-1)|v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i| - \sum_{\substack{i,j=1 \\ i\neq j}}^{k} |v_i\rangle\langle v_j| \otimes |w_i\rangle\langle w_j| \\
&= \sum_{\substack{i,j=1 \\ i\neq j}}^{k} \Big(|v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i| - |v_i\rangle\langle v_j| \otimes |w_i\rangle\langle w_j| \Big) \\
&= \sum_{i=1}^{k} \sum_{j=i+1}^{k} \Big(|v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i| - |v_i\rangle\langle v_j| \otimes |w_i\rangle\langle w_j| \\
&\quad - |v_j\rangle\langle v_i| \otimes |w_j\rangle\langle w_i| + |v_j\rangle\langle v_j| \otimes |w_j\rangle\langle w_j| \Big) \\
&= \sum_{i=1}^{k} \sum_{j=i+1}^{k} \big(|v_i\rangle \otimes |w_i\rangle - |v_j\rangle \otimes |w_j\rangle \big)\big(\langle v_i| \otimes \langle w_i| - \langle v_j| \otimes \langle w_j| \big) \\
&\geq 0
\end{aligned}$$

□

We thus see that a necessary condition for $SN(\rho) \leq k$ is that $(id_n \otimes \Phi_k)(\rho) \geq 0$. We can restate this as the following *Generalized Reduction Criterion*:

Theorem 2.4 (Generalized Reduction Criterion)**.** *If $SN(\rho) \leq k$ then $k\rho_2 \otimes I \geq \rho$ and $kI \otimes \rho_1 \geq \rho$.*

As indicated by its name, the Generalized Reduction Criterion generalizes the standard reduction criterion of Horodecki and Horodecki [6]. The following corollary is trivial and thus its proof is omitted.

Corollary 2.5. *If the maximum eigenvalue of $\rho \in M_m \otimes M_n$ is strictly greater than k times the maximum eigenvalue of either of its reduced density matrices, then its Schmidt Number is at least $k+1$.*

2.6. **Relationship with k-Positive Maps.** As was seen in the previous section, $(id_n \otimes \Phi_k)(\rho) \geq 0$ for all $\rho \in M_n \otimes M_n$ with $SN(\rho) \leq k$, where $\Phi_k(\sigma) = k\mathrm{Tr}(\sigma) \cdot I_n - \sigma$. The following theorem shows that this is no coincidence; it occurs precisely because Φ_k is a k-positive map (a fact that is well-known to operator algebraists).

Recall at this point that the set of k-positive linear maps is denoted by $\mathcal{P}_k$.

Theorem 2.6 (k-Positive Maps)**.** *Let $\Phi : M_n \mapsto M_m$ be a linear map. Then $\Phi \in \mathcal{P}_k$ if and only if*

$$(id_n \otimes \Phi)(\rho) \geq 0 \quad \forall\, \rho \in M_n \otimes M_n \text{ with } SN(\rho) \leq k.$$

The above theorem is well-known in quantum information, and it can be reworked slightly to show that the exact same inequality characterizes the Schmidt Number of density matrices in terms of k-positive maps:

Theorem 2.7 (Terhal and Horodecki [14])**.** *Let $\rho \in M_n \otimes M_n$ be a density matrix. Then $SN(\rho) \leq k$ if and only if*

$$(id_n \otimes \Phi)(\rho) \geq 0 \quad \forall\, \Phi \in \mathcal{P}_k.$$

A well-known special case of Terhal and Horodecki's Theorem arises when $k = 1$, and it says that ρ is separable if and only if $(id_n \otimes \Phi)(\rho) \geq 0$ for all positive maps Φ. The original proof of this result is quite technical, so it will not be proved here. Instead it will come as a corollary of the much more general Theorem 4.5, which will be presented and proved in Section 4.4.

3. Partially Entanglement Breaking Maps

3.1. **Definition.** By recalling that the Schmidt Number of a state equals 1 if and only if that state is separable, the following can be seen as a generalization of the well-studied entanglement breaking maps.

Definition 3.1 (Partially Entanglement Breaking Maps [3])**.** *A completely positive map $\Phi : M_n \mapsto M_m$ is said to be k-partially entanglement breaking (k-PEB) if $SN((id_n \otimes \Phi)(\rho)) \leq k$ for all $0 \leq \rho \in M_n \otimes M_n$. The set of k-PEB maps will be denoted by $\mathcal{B}_k$.*

It will be seen shortly that the notion of a map being k-partially entanglement breaking generalizes the notion of the map being entanglement breaking in the exact same way that the Schmidt Number of a state generalizes whether or not that state is separable. Also, as with the Schmidt Number for density matrices, there are two special cases to note here:

- Entanglement breaking maps are exactly the 1-PEB maps.
- The set of $\min\{m, n\}$-PEB maps is exactly the set of all completely positive maps.

3.2. **Characterization.** In order to prove relationships between k-PEB maps and other well-known families of maps and operators, the following useful fact will often be used: $\Phi \in \mathcal{B}_k$ if and only if it can be written in the form

$$\Phi(\rho) = \sum_p \sum_{q,r=1}^{k} \langle v_{p,q}|\rho|v_{p,r}\rangle |w_{p,q}\rangle\langle w_{p,r}|. \tag{3}$$

The above characterization of k-PEB maps will be proved as part of the upcoming theorem. Recall that $|e\rangle$ denotes the maximally entangled state $|e\rangle = \sum_{j=1}^{n} |e_j\rangle \otimes |e_j\rangle$, where $\{|e_j\rangle\}$ is the standard orthonormal basis of $\mathbb{C}^n$. Thus, $(id_n \otimes \Phi)(|e\rangle\langle e|)$ is the Choi matrix (which we will often denote by C_Φ) of Φ.

We could equivalently replace $|e\rangle$ with any other maximally-entangled pure state at any point throughout this paper - the vector $|e\rangle$ is used for simplicity and its direct connection with the Choi matrix.

Theorem 3.2 (PEB Characterization)**.** *Let $\Phi : M_n \mapsto M_m$ be a completely positive linear map. Then the following are equivalent:*

a) $\Phi \in \mathcal{B}_k$.
b) $SN((id_n \otimes \Phi)(|e\rangle\langle e|)) \leq k$.
c) Φ *can be written in the form* (3) *above.*
d) Φ *can be written in operator sum form using Kraus operators of rank at most* k.
e) $\Psi \circ \Phi$ *is completely positive for all* k*-positive maps* Ψ.
f) $\Phi \circ \Psi$ *is completely positive for all* k*-positive maps* Ψ.

Proof. The implication a) $\Rightarrow$ b) follows trivially from the definition of k-PEB maps. To see b) $\Rightarrow$ c) note that

$$(id_n \otimes \Phi)(|e\rangle\langle e|) = \sum_{i,j} |e_i\rangle\langle e_j| \otimes \Phi(|e_i\rangle\langle e_j|),$$

and this state must have Schmidt Number at most k. Thus there must exist vectors $\{|\psi_p\rangle\}$ so that $|\psi_p\rangle = \sum_{q=1}^{k} |v_{p,q}\rangle \otimes |w_{p,q}\rangle$ and

$$\begin{aligned}\sum_{i,j} |e_i\rangle\langle e_j| \otimes \Phi(|e_i\rangle\langle e_j|) &= \sum_{p} |\psi_p\rangle\langle\psi_p| \\ &= \sum_{p}\sum_{q,r=1}^{k} |v_{p,q}\rangle\langle v_{p,r}| \otimes |w_{p,q}\rangle\langle w_{p,r}|.\end{aligned}$$

Now let Ω be a map of the form (3) defined by

$$\Omega(\rho) = \sum_{p}\sum_{q,r=1}^{k} \overline{\langle v_{p,q}|}\rho\overline{|v_{p,r}\rangle}|w_{p,q}\rangle\langle w_{p,r}|,$$

where $\overline{|v_{p,r}\rangle} = \sum_j \langle v_{p,r}|e_j\rangle|e_j\rangle$. Then by recalling that $|v_{p,r}\rangle = \sum_j \langle e_j|v_{p,r}\rangle|e_j\rangle$ it is not difficult to verify that

$$\begin{aligned}(id_n \otimes \Omega)(|e\rangle\langle e|) &= \sum_{i,j,p}\sum_{q,r=1}^{k} |e_i\rangle\langle e_j| \otimes \overline{\langle v_{p,q}|e_i\rangle\langle e_j|v_{p,r}\rangle}|w_{p,q}\rangle\langle w_{p,r}| \\ &= \sum_{p}\sum_{q,r=1}^{k} |v_{p,q}\rangle\langle v_{p,r}| \otimes |w_{p,q}\rangle\langle w_{p,r}|.\end{aligned}$$

Thus $(id_n \otimes \Phi)(|e\rangle\langle e|) = (id_n \otimes \Omega)(|e\rangle\langle e|)$ and so it follows that $\Phi = \Omega$ because linear maps are determined by their action on the orthonormal basis $|e_i\rangle\langle e_j|$.

To see c) $\Rightarrow$ d) simply note that

$$\Phi(\rho) = \sum_p \sum_{q,r=1}^k \langle v_{p,q}|\rho|v_{p,r}\rangle |w_{p,q}\rangle\langle w_{p,r}|$$
$$= \sum_{p=1} \Big[\sum_{q=1}^k |w_{p,q}\rangle\langle v_{p,q}| \Big] \rho \Big[\sum_{r=1}^k |v_{p,r}\rangle\langle w_{p,r}| \Big],$$

and each $\big[\sum_{q=1}^k |w_{p,q}\rangle\langle v_{p,q}| \big]$ has rank at most k.

To see d) $\Rightarrow$ a), recall that we can write a rank 1 density matrix $\rho \in M_n \otimes M_n$ as $\rho = |x\rangle\langle x|$, where $|x\rangle = \sum_i |y_i\rangle \otimes |z_i\rangle$. Thus,

$$(id_n \otimes \Phi)(\rho) = \sum_{i,j} |y_i\rangle\langle y_j| \otimes \sum_p \sum_{q,r=1}^k |w_{p,q}\rangle\langle v_{p,q}|z_i\rangle\langle z_j|v_{p,r}\rangle\langle w_{p,r}|$$
$$= \sum_{i,j,p} \sum_{q,r=1}^k \langle v_{p,q}|z_i\rangle\langle z_j|v_{p,r}\rangle \, |y_i\rangle\langle y_j| \otimes |w_{p,q}\rangle\langle w_{p,r}|$$
$$= \sum_p \Big[\sum_{q=1}^k |x_{p,q}\rangle \otimes |w_{p,q}\rangle \Big] \Big[\sum_{r=1}^k \langle x_{p,r}| \otimes \langle w_{p,r}| \Big],$$

where $|x_{p,q}\rangle = \sum_i \langle v_{p,q}|z_i\rangle |y_i\rangle$. The above state has Schmidt Number at most k, so the implication follows from linearity.

We have established the equivalence of conditions a) through d).

The equivalence of a) and e) comes as an immediate corollary of Theorem 2.7 of Terhal and Horodecki, described earlier.

The equivalence of e) and f) comes from the following three simple facts: Ψ is k-positive if and only if $\Psi^\dagger$ is k-positive, Φ is k-PEB if and only if $\Phi^\dagger$ is k-PEB (which can be seen as a corollary of the equivalence of a) and d)), and $(\Psi \circ \Phi)^\dagger = \Phi^\dagger \circ \Psi^\dagger$. □

The various equivalences of the above characterization were each proved in [3] and can be regarded as a generalization of the characterization of entanglement breaking maps of [7]. The proof above is presented for completeness because it is elementary, and because it establishes the equivalence of a) through d) in a constructive manner.

Conditions e) and f) provide some motivation for the interpretation of k-PEB maps as maps that are somehow "more" positive than generic CP maps, in a way that is complementary to how k-positive maps are "less" positive than CP maps.

3.3. **Results.** The following theorem generalizes a result of Horodecki, Shor and Ruskai [7] that says that entanglement breaking channels can't be represented with fewer than n Kraus operators. The intuitive understanding of the theorem is that k-PEB maps can be interpreted as channels with lots of noise, because they destroy entanglement. Since they represent channels with lots of noise, it makes sense that their Choi-Kraus representation should have multiple error operators.

Theorem 3.3. *Let $k \geq 1$ and suppose that $\Phi : M_n \mapsto M_m$ is a k-PEB quantum channel. Then Φ can not be represented with fewer than $\lceil n/k \rceil$ Kraus operators.*

Proof. This proof will proceed by proving the contrapositive of the given statement. Suppose that the fewest Kraus operators that a quantum channel Φ can be written with is r. Then $r = rank(C_\Phi)$ by a result of Choi [2].

We also know that $Tr_2(C_\Phi) = I_n$ since Φ is trace-preserving. Since C_Φ has rank $r < \frac{n}{k}$ and has trace equal to n, it follows that $\lambda_{max} > k$, where λ_{max} is the largest eigenvalue of C_Φ. If follows from the Generalized Reduction Criterion that $SN(C_\Phi) > k$ and so the characterization theorem tells us that Φ is not k-PEB. □

4. Cones of Positive Maps

4.1. **Jamiolkowski Isomorphism.** The Jamiolkowski Isomorphism [9] is a way of associating with every $\Phi \in \mathcal{L}(M_n, M_m)$ a unique $C_\Phi \in M_{nm} \cong M_n(M_m)$. As the notation suggests, the isomorphism indeed just takes Φ to its Choi matrix, $C_\Phi := (id_n \otimes \Phi)(|e\rangle\langle e|)$. It is easy to see that this is really an isomorphism because $(id_n \otimes \Phi)(|e\rangle\langle e|)$ is just an $n \times n$ block matrix of the action of Φ on each of the standard matrix units $|e_i\rangle\langle e_j|$.

4.2. **Cones and Dual Cones.** Notice that if $\lambda \in \mathbb{R}_+$ and $\Phi \in \mathcal{CP}$ then $\lambda \cdot \Phi \in \mathcal{CP}$. Thus, the set $C_{\mathcal{CP}}$ of Choi matrices of completely positive maps forms a cone in M_{nm} (indeed, $C_{\mathcal{CP}} = M_{nm}^+$). Similarly, it is easy to verify that the sets $C_{\mathcal{P}_k}$ and $C_{\mathcal{B}_k}$ are also cones in M_{nm}. These cones are the primary motivation for the definition and investigation of *right-CP invariant cones* in the next section.

Given a cone $\mathcal{C} \subseteq M_{nm}^{sa}$ of self-adjoint matrices, one can define its *dual cone* in the usual way:

$$\mathcal{C}^d \equiv \big\{ D \in M_{nm}^{sa} \,|\, \langle C|D\rangle_{HS} \equiv \mathrm{Tr}(CD) \geq 0 \;\; \forall\, C \in \mathcal{C} \big\}.$$

Given a cone $\mathcal{S} \subseteq \mathcal{L}(M_n, M_m)$, we may use the Choi-Jamiolkowski Isomorphism to find the cone $C_{\mathcal{S}} \subseteq M_{nm}$, use the dual cone definition above to find $C_{\mathcal{S}}^d$, and use the Choi-Jamiolkowski Isomorphism again to find another cone of linear maps, which we will denote by $\mathcal{S}^D$:

$$\mathcal{S}^D \equiv \big\{ \Phi \in \mathcal{L}(M_n, M_m) \,|\, C_\Phi \in C_{\mathcal{S}}^d \big\}.$$

We will call $\mathcal{S}^D$ the *dual cone* of $\mathcal{S}$. It will be clear from the context whether we are talking about the dual cone of matrices or the dual cone of linear maps, though this distinction is not usually very important due to the Jamiolkowski Isomorphism.

4.3. **The Dual Cone of k-Positive Maps.** In the $k = 1$ case, it is well known [13] that $\Phi \in \mathcal{P}_1$ if and only if $(\langle x| \otimes \langle y|)C_\Phi(|x\rangle \otimes |y\rangle) \geq 0$ for all $|x\rangle \in \mathbb{C}^n, |y\rangle \in \mathbb{C}^m$.

The above fact can be used to show that $\mathcal{P}_1^D = \mathcal{B}_1$ and $\mathcal{B}_1^D = \mathcal{P}_1$. That is, the entanglement breaking maps are the dual cone of the positive maps, and vice-versa. The same result holds for higher k, as has been hinted at by the recent literature on the subject [10, 12].

It was proved in [10] that Φ is k-positive if and only if $\langle v|C_\Phi|v\rangle \geq 0$ for all vectors $|v\rangle$ with Schmidt Rank at most k – this fact can also be seen as a corollary of Terhal and Horodecki's theorem from earlier. We will prove it here for completeness.

Theorem 4.1. *$\Phi \in \mathcal{P}_k$ if and only if $\langle v|C_\Phi|v\rangle \geq 0$ for all vectors $|v\rangle \in \mathbb{C}^{nm}$ with Schmidt Rank at most k.*

Proof. The proof will simply make use of the fact that $(id_k \otimes \Phi)$ is positive if and only if $((\langle x| \otimes \langle y|)C_{id_k\otimes\Phi}(|x\rangle \otimes |y\rangle)) \geq 0$ for all $|x\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n, |y\rangle \in \mathbb{C}^k \otimes \mathbb{C}^m$.

First write $|x\rangle = \sum_{i=1}^k |e_i\rangle \otimes |x_i\rangle$ and $|y\rangle = \sum_{i=1}^k |e_i\rangle \otimes |y_i\rangle$. Then

$$|x\rangle \otimes |y\rangle = \sum_{i,j=1}^{k} |e_i\rangle \otimes |x_i\rangle \otimes |e_j\rangle \otimes |y_j\rangle$$

and after some simplification we see that

$$\begin{aligned}
\Phi \in \mathcal{P}_k &\iff ((\langle x| \otimes \langle y|)C_{id_k\otimes\Phi}(|x\rangle \otimes |y\rangle)) \geq 0 \quad \forall |x\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n, |y\rangle \in \mathbb{C}^k \otimes \mathbb{C}^m \\
&\iff \sum_{i,l=1}^{k}\sum_{r,s=1}^{n} ((\langle x_l| \otimes \langle y_l|)|e_r\rangle\langle e_s| \otimes \Phi(|e_r\rangle\langle e_s|)(|x_i\rangle \otimes |y_j\rangle)) \geq 0 \quad \forall \{|x_l\rangle\}, \{|y_l\rangle\} \\
&\iff \sum_{i=1}^{k}\Big[\langle x_i| \otimes \langle y_i|\Big]C_\Phi \sum_{i=1}^{k}\Big[|x_i\rangle \otimes |y_i\rangle\Big] \geq 0 \quad \forall \{|x_i\rangle\} \in \mathbb{C}^n, \{|y_i\rangle\} \in \mathbb{C}^m \\
&\iff \langle v|C_\Phi|v\rangle \geq 0 \quad \forall |v\rangle \text{ with Schmidt Rank at most } k.
\end{aligned}$$

□

Similarly, we saw earlier (Theorem 3.2) that $\Phi \in \mathcal{B}_k$ if and only if $C_\Phi = \sum_j |v_j\rangle\langle v_j|$, where each $|v_j\rangle$ has Schmidt Rank at most k.

Putting these results together, we thus find that the dual cone of the k-PEB maps are exactly the k-positive maps:

$$\begin{aligned}
&\Phi \in \mathcal{B}_k^D \\
&\iff \mathrm{Tr}(C_\Phi \rho) \geq 0 \quad \forall \rho \in C_{\mathcal{B}_k} \\
&\iff \mathrm{Tr}(C_\Phi |v\rangle\langle v|) \geq 0 \quad \forall |v\rangle \text{ with Schmidt Rank at most } k \\
&\iff \langle v|C_\Phi|v\rangle \geq 0 \quad \forall |v\rangle \text{ with Schmidt Rank at most } k \\
&\iff \Phi \in \mathcal{P}_k.
\end{aligned}$$

Indeed, because $C_{\mathcal{B}_k}$ is a closed, convex cone it follows that $\mathcal{P}_k^D = \mathcal{B}_k$ as well.

4.4. **Other Cones of Positive Maps.** We have seen several ways of characterizing positive linear maps $\Phi : M_n \mapsto M_m$ by looking at cones on which $(id_n \otimes \Phi)$ is positive. Here we will see how these results extend to cones of positive maps other than PEB or k-positive maps. We begin with a definition and some basic observations about the cones of linear maps that we have dealt with so far.

Definition 4.2. *Let $\mathcal{S} \subseteq \mathcal{L}(M_n, M_m)$ be a set of positive linear maps. $\mathcal{S}$ will be said to be a* right CP-invariant cone *if $\Phi \circ \Psi \in \mathcal{S}$ whenever $\Phi \in \mathcal{S}$ and $\Psi \in \mathcal{CP}$.*

- The definition of a *left CP-invariant cone* should be obvious.
- It is clear that $\mathcal{P}_k$ is both left and right CP-invariant.
- It is clear that $\mathcal{B}_k$ is right CP-invariant. The fact that it is left CP-invariant comes from Theorem 2.3.
- In [11] a *mapping cone* was defined to be a cone that is both *left* and *right CP-invariant*, though this terminology will not be used in this paper.

Before delving too deep into results concerning right CP-invariant cones, it will be useful to formally present some useful (albeit trivial) properties.

Lemma 4.3. *$\mathcal{S}$ is a right CP-invariant cone if and only if $\mathcal{S}^\dagger \equiv \{\Phi^\dagger | \Phi \in \mathcal{S}\}$ is a left CP-invariant cone.*

Proof. Simply notice that $\Psi \in \mathcal{CP} \Leftrightarrow \Psi^\dagger \in \mathcal{CP}$ and $(\Phi \circ \Psi)^\dagger = \Psi^\dagger \circ \Phi^\dagger$. □

The above lemma shows that all of the following results about right CP-invariant cones can easily be modified to be stated in terms of left CP-invariant cones.

Lemma 4.4. *Let $\mathcal{S}$ be a right CP-invariant cone. Then the following are equivalent:*

a) $\Phi \in \mathcal{S}$
b) $(id_n \otimes \Phi)(\rho) \in C_{\mathcal{S}}$ *for all* $0 \leq \rho \in M_n \otimes M_n$

Furthermore, if a set $\mathcal{S}' \subseteq \mathcal{L}(M_n, M_m)$ is such that a) $\Rightarrow$ b), then $\mathcal{S}'$ must be a right CP-invariant cone.

Proof. The implication b) $\Rightarrow$ a) is trivial because we can take $\rho = |e\rangle\langle e|$.

To see a) $\Rightarrow$ b), note that because $\mathcal{S}$ is a right CP-invariant cone, we have that $\Phi \circ \Psi \in \mathcal{S}$ for all $\Psi \in \mathcal{CP}$. Thus $(id_n \otimes \Phi \circ \Psi)(|e\rangle\langle e|) = (id_n \otimes \Phi)(C_\Psi) \in C_{\mathcal{S}}$ for all $\Psi \in \mathcal{CP}$. The result then comes from Choi's result [2] that $\Psi \in \mathcal{CP} \Leftrightarrow C_\Psi \geq 0$ and the Jamiolkowski Isomorphism.

The final claim comes using the exact same reasoning used to establish the implication a) $\Rightarrow$ b) in reverse. □

In terms of k-PEB maps, the above lemma gives us exactly the equivalence of a) and b) in the PEB Characterization Theorem 3.2.

Theorem 4.5. *Let $\mathcal{S} \subseteq \mathcal{L}(M_n, M_m)$ be a cone and consider the following three properties:*

a) $\rho \in C_{\mathcal{S}}^d$
b) $\langle e|(id_n \otimes \Phi^\dagger)(\rho)|e\rangle \geq 0 \quad \forall\, \Phi \in \mathcal{S}$
c) $(id_n \otimes \Phi^\dagger)(\rho) \geq 0 \quad \forall\, \Phi \in \mathcal{S}$

Then

(1) a) $\Leftrightarrow$ b), and
(2) If $\mathcal{S}$ is right CP-invariant then a) and c) are equivalent.

Proof. To see (1), note that

$$\begin{aligned}
&\langle e|(id_n \otimes \Phi^\dagger)(\rho)|e\rangle \geq 0 \quad \forall\, \Phi \in \mathcal{S} \\
\Longleftrightarrow\ &\mathrm{Tr}(|e\rangle\langle e|(id_n \otimes \Phi)^\dagger(\rho)) \geq 0 \quad \forall\, \Phi \in \mathcal{S} \\
\Longleftrightarrow\ &\mathrm{Tr}((id_n \otimes \Phi)(|e\rangle\langle e|)\rho) \geq 0 \quad \forall\, \Phi \in \mathcal{S} \\
\Longleftrightarrow\ &\mathrm{Tr}(C_\Phi \rho) \geq 0 \quad \forall\, C_\Phi \in C_{\mathcal{S}} \\
\Longleftrightarrow\ &\rho \in C_{\mathcal{S}}^d.
\end{aligned}$$

The implication c) $\Rightarrow$ b) is trivial. To see the reverse implication, note that if $\mathcal{S}$ is a right CP-invariant cone, we can replace $\Phi^\dagger$ with $\Psi^\dagger \circ \Phi^\dagger$ where $\Psi \in \mathcal{CP}$ is arbitrary. We thus have that

$$\begin{aligned}
&\langle e|(id_n \otimes \Psi^\dagger \circ \Phi^\dagger)(\rho)|e\rangle \geq 0 \quad \forall\, \Phi \in \mathcal{S}, \Psi \in \mathcal{CP} \\
\Longrightarrow\ &\Big(\sum_{j=1}^n \langle e_j| \otimes \langle e_j|X\Big)(id_n \otimes \Phi^\dagger)(\rho)\Big(\sum_{j=1}^n |e_j\rangle \otimes X^\dagger|e_j\rangle\Big) \geq 0 \quad \forall\, \Phi \in \mathcal{S}, \forall X \in M_{n,m} \\
\Longrightarrow\ &\Big(\sum_{j=1}^n \langle e_j| \otimes \langle x_j|\Big)(id_n \otimes \Phi^\dagger)(\rho)\Big(\sum_{j=1}^n |e_j\rangle \otimes |x_j\rangle\Big) \geq 0 \quad \forall\, \Phi \in \mathcal{S}, \forall X \in M_{n,m},
\end{aligned}$$

where $\langle x_j|$ is the j^{th} row of X. Since any $|x\rangle$ can be written in the form $|x\rangle = \sum_{j=1}^n |e_j\rangle \otimes |x_j\rangle$, (2) established. □

If we take $\mathcal{S} = \mathcal{P}_k$ then the equivalence of a) and c) in the above theorem gives us exactly Terhal and Horodecki's Theorem.

It will be useful to "flip" the roles of operators and linear maps in Theorem 4.5, and the following Lemma is the stepping stone that we need.

Lemma 4.6. *Let* $\Phi, \Psi \in \mathcal{L}(M_n, M_m)$*. Then* $(id_n \otimes \Psi^\dagger)(C_\Phi) \geq 0 \Leftrightarrow (id_n \otimes \Phi^\dagger)(C_\Psi) \geq 0$.

Proof.

$$\begin{aligned}
&(id_n \otimes \Psi^\dagger)(C_\Phi) \geq 0 \\
\Longleftrightarrow\ &(id_n \otimes \Psi^\dagger \circ \Phi)(|e\rangle\langle e|) \geq 0 \\
\Longleftrightarrow\ &\Psi^\dagger \circ \Phi \in \mathcal{CP} \\
\Longleftrightarrow\ &\Phi^\dagger \circ \Psi \in \mathcal{CP} \\
\Longleftrightarrow\ &(id_n \otimes \Phi^\dagger \circ \Psi)(|e\rangle\langle e|) \geq 0 \\
\Longleftrightarrow\ &(id_n \otimes \Phi^\dagger)(C_\Psi) \geq 0
\end{aligned}$$

□

Corollary 4.7. *Let* $\mathcal{S} \subseteq \mathcal{L}(M_n, M_m)$ *be a cone and consider the following three properties:*
a) $\Psi \in \mathcal{S}^D$
b) $\langle e|(id_n \otimes \Psi^\dagger)(\rho)|e\rangle \geq 0 \quad \forall\, \rho \in C_{\mathcal{S}}$
c) $(id_n \otimes \Psi^\dagger)(\rho) \geq 0 \quad \forall\, \rho \in C_{\mathcal{S}}$

Then

(1) a) $\Leftrightarrow$ b), and
(2) If $\mathcal{S}$ is right CP-invariant then a) and c) are equivalent.

Proof. (1) is proved in the same way as Theorem 4.5. (2) follows from the analogous results in Theorem 4.5 via Lemma 4.6. □

If we take $\mathcal{S} = \mathcal{B}_k$ then the equivalence of a) and c) in the above corollary gives us exactly Theorem 2.6.

Corollary 4.8. *Let $\mathcal{S} \subseteq \mathcal{L}(M_n, M_m)$ be a right CP-invariant cone. Then the following are equivalent:*
a) $\Psi \in \mathcal{S}^D$
b) $\Phi^\dagger \circ \Psi \in \mathcal{CP}$ for all $\Phi \in \mathcal{S}$

Proof. The result comes from associating $\rho \in C^d_{\mathcal{S}}$ with a map $\Psi \in \mathcal{S}^D$ in Theorem 4.5. □

By taking $\mathcal{S} = \mathcal{P}_k$ in the above corollary, we get the equivalence of a), e) and f) in the PEB Characterization Theorem 3.2.

5. Conclusions

We have built up from the basic ideas of Schmidt Rank and Schmidt Number to examine partially entanglement breaking maps and generalize many well-known concepts and results involving separable states, positive maps, and entanglement breaking maps. We have also seen that many of the important properties of partially entanglement breaking maps follow precisely because they form a right CP-invariant cone, and their interplay with the k-positive maps follows because they are dual cones of each other. In this way we have generalized many of the characterizations of separable states, positive maps and entanglement breaking maps to much more general cones of states and maps.

Many natural cones of quantum maps are not right CP-invariant – trace-preserving maps, unital maps, randomized unitary channels, and so on. Thus a natural next step is to extend the results of Section 4.4, perhaps to what would be called *right unitarily-invariant cones*, and explore the implications that follow.

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