PARTIALLY ENTANGLEMENT BREAKING MAPS AND RIGHT CP-INVARIANT CONES

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Abstract. There are many results in quantum computing that give ways of characterizing classes of density matrices in terms of other classes of linear maps, and vice-versa – Horodecki's theorem for positive entanglement witnesses, Terhal and Horodecki's theorem for density matrices with Schmidt Number at most k, and so on. The goal of this talk is to determine for which sets of linear maps and density matrices these types of relationships hold. Along the way a characterization of Partially Entanglement Breaking maps will be given, and it will be shown that many of their well-known characterizations follow simply because they form a cone that is invariant under right composition with CP maps – a fact that is trivial to check.

1. Introduction and Preliminaries

1.1. **Intended Audience.** This talk is aimed at an audience with a strong grasp of linear algebra and matrix or operator theory. Knowledge of standard linear algebra results such as the Singular Value Decomposition will be an asset.

Some basic quantum computing knowledge will be required as well, such as the definition a of density matrix and a completely positive map. A single course in quantum information should provide sufficient background to the reader.

1.2. **Notation.** Throughout this talk I will focus largely on the space of $m \times n$ complex matrices, denoted $M_{m,n}$. When $m = n$, this will be abbreviated as M_n.

Dirac notation such as $|\phi\rangle$ will be used for vectors in $\mathbb{C}^n$. $|\phi\rangle$ will always be a column vector, and we define $\langle\phi| \equiv |\phi\rangle^\dagger$ to be the adjoint of $|\phi\rangle$.

The inner product of vectors $|\phi\rangle, |\psi\rangle \in \mathbb{C}^n$ will be denoted by $\langle\psi|\phi\rangle$ and it is worth noting that because of how this notation is defined the inner product is linear in its *second* variable and conjugate linear in its *first* variable.

1.3. **Schmidt Decomposition Theorem.** As will be seen in its proof, the Schmidt Decomposition Theorem is essentially the Singular Value Decomposition in disguise. For the purposes of this talk, it is valuable because it will lead us to the definition of the Schmidt Rank of a pure state.

Theorem 1.1 (Schmidt Decomposition)**.** *For any $|\phi\rangle \in \mathbb{C}_m \otimes \mathbb{C}_n$ there exist orthonormal sets of vectors $\{|u_1\rangle, |u_2\rangle, \ldots, |u_m\rangle\} \subset \mathbb{C}_m$ and $\{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\} \subset \mathbb{C}_n$ such that*

$$|\phi\rangle = \sum_{j=1}^{\min\{m,n\}} \alpha_j |u_j\rangle \otimes |v_j\rangle$$

for some non-negative real constants $\{\alpha_j\}$.

Proof. Assume that $n \leq m$, as it will be clear how to modify the proof if the opposite inequality holds. Note that for any $|\psi_1\rangle \in \mathbb{C}_m$ and $|\psi_2\rangle \in \mathbb{C}_n$, we can isomorphically associate $|\psi_1\rangle \otimes |\psi_2\rangle \in \mathbb{C}_m \otimes \mathbb{C}_n$ with $|\psi_1\rangle\overline{\langle\psi_2|} \in M_{m,n}$ (in fact, this isomorphism is isometric if the norm we put on $M_{m,n}$ is the Frobenius norm). The Singular Value Decomposition tells us that if $A \in M_{m,n}$ then there exist unitaries $U \in M_m$, $V \in M_n$, and a positive semidefinite diagonal matrix $D \in M_m$ such that

$$A = U \begin{bmatrix} D \\ 0 \end{bmatrix} V.$$

∎

If we expand out the previous matrix multiplication then we see that

$$A = \sum_{j=1}^{n} \alpha_j |u_j\rangle\langle v_j|,$$

where α_j is the j^{th} diagonal entry of D, $|u_j\rangle$ is the j^{th} column of U, and $\langle v_j|$ is the j^{th} row of V. Since U and V are both unitaries, simply associating A back with the vector

$$\sum_{j=1}^{n} \alpha_j |u_j\rangle \otimes \overline{|v_j\rangle} \in \mathbb{C}^m \otimes \mathbb{C}^n$$

completes the proof.

2. Schmidt Rank and Schmidt Number

2.1. **Schmidt Rank.** In the Schmidt Decomposition of a vector $|\phi\rangle$, the least number of terms required in the summation is known as the *Schmidt Rank* of $|\phi\rangle$.

By following along through the proof, we see that the Schmidt Rank of $|\phi\rangle$ is equal to the number of non-zero singular values of the matrix $A_\phi \in M_{m,n}$ to which $|\phi\rangle$ is associated. Thus, the Schmidt Rank of $|\phi\rangle$ is also equal to the rank of A_ϕ.

Further, since each $|u_j\rangle\langle v_j| \in M_{m,n}$ has rank 1, we see that even if we remove the requirement that the sets $\{|u_1\rangle, \ldots, |u_m\rangle\}$ and $\{|v_1\rangle, \ldots, |v_n\rangle\}$ be orthonormal, it is impossible to write $|\phi\rangle$ as the sum of fewer elementary tensors.

In the language of quantum computing, this gives us a way to compute a number that roughly measures the "amount of entanglement" contained within a pure state.

- Separable pure states are represented exactly by the vectors with Schmidt Rank equal to 1.
- If $m = n$ then maximally-entangled pure states have Schmidt Rank equal to n (although the converse does not hold).

We would like to extend this notion to density matrices so that we can define a modified Schmidt Rank for density matrices (that is, pure as well as mixed states).

2.2. **Schmidt Number.** Recall that density matrices are positive-semidefinite and thus can be written in the following form:

$$\rho = \sum_{j=1} |v_j\rangle\langle v_j| \in M_m \otimes M_n. \tag{1}$$

Definition 2.1 (Schmidt Number [12])**.** *The* Schmidt Number *of a density matrix ρ (denoted by $SN(\rho)$) is the least natural number k such that it can be written in the form* (1) *using vectors $|v_j\rangle$ with Schmidt Rank not greater than k.*

It is easy to see that for a pure state $|\phi\rangle$, the Schmidt Rank of $|\phi\rangle$ coincides with the Schmidt Number of $|\phi\rangle\langle\phi|$, as we would hope. Also, the Schmidt Number of a density matrix still tells us roughly "how entangled" that state is.

- Separable states are represented by density matrices of the form $\rho = \sum_j \sigma_j \otimes \tau_j$, where each $\sigma_j, \tau_j \geq 0$. These are exactly the density matrices with Schmidt Number equal to 1.
- If $m = n$ then maximally-entangled states have Schmidt Number equal to n (although the converse does not hold).

Furthermore, we have the very desirable (and intuitive) property that applying a quantum operation to one "piece" of a given state ρ can not increase its Schmidt Number. This is the content of the following theorem.

Theorem 2.2. *Let $\rho \in M_m \otimes M_n$ be a density matrix and let $\Phi : M_n \mapsto M_k$ be a completely positive map. Then*

$$SN((id_m \otimes \Phi)(\rho)) \leq SN(\rho).$$

Proof. Recall that a result of Choi [2] says that Φ can be written as $\Phi(X) = \sum_l A_l X A_l^\dagger$ for some family $\{A_l\} \in M_{k,n}$. Assume that $\rho = |\psi\rangle\langle\psi|$ is a pure state with $SN(\rho) = k$ and write $|\psi\rangle = \sum_{i=1}^k |v_i\rangle \otimes |w_i\rangle$ so that

$$\begin{aligned}(id_m \otimes \Phi)(\rho) &= \sum_{i,j=1}^k |v_i\rangle\langle v_j| \otimes \Phi(|w_i\rangle\langle w_j|) \\ &= \sum_{i,j=1}^k |v_i\rangle\langle v_j| \otimes \sum_l A_l|w_i\rangle\langle w_j|A_l^\dagger \\ &= \sum_l \Big(\sum_{i=1}^k |v_i\rangle \otimes A_l|w_i\rangle\Big)\Big(\sum_{i=1}^k |v_i\rangle \otimes A_l|w_i\rangle\Big)^\dagger,\end{aligned}$$

which is a density matrix with Schmidt Number no more than k. The result follows simply by linearity and the fact that pure states span all density matrices. ∎

In slightly more generality, it is well-known that the Schmidt Number of a state is non-increasing under local quantum operations and classical communication [12].

2.3. **Generalized Separability Criteria.** We saw earlier that determining the Schmidt Rank of a pure state isn't a particularly challenging task. The situation is much different for determining the Schmidt Number of an arbitrary state, however – determining the Schmidt Number of an arbitrary state (and even just whether the state is separable or not) is an NP-hard problem.

Despite the hardness of finding the exact Schmidt Number of a given state ρ, we will soon see how we can bound $SN(\rho)$ from below by examining the action of maps of the form $(id_n \otimes \Phi)$ on ρ. It will be illustrative to start with a simple example.

Let $k \in \mathbb{N}$ and consider the map $\Phi_k : M_n \mapsto M_n$ defined by $\Phi_k(\rho) = k\mathrm{Tr}(\rho) \cdot I_n - \rho$. We will now see that if $\rho \in M_n \otimes M_n$ is such that $SN(\rho) \leq k$ then $(id_n \otimes \Phi)(\rho) \geq 0$, even though it is easy to check that Φ_k is not completely positive if $k < n$.

Proof that $SN(\rho) \leq k \Rightarrow (id_n \otimes \Phi_k)(\rho) \geq 0$. First notice that, due to linearity, it is enough to require that $(id_n \otimes \Phi)$ be positive on pure states with Schmidt Rank at most k. Thus, consider an arbitrary such pure state:

$$\rho = |\psi\rangle\langle\psi| = \sum_{i,j=1}^{k} |v_i\rangle\langle v_j| \otimes |w_i\rangle\langle w_j|,$$

where $\{|v_j\rangle\}, \{|w_j\rangle\} \subseteq \mathbb{C}^n$ are orthogonal sets (and the $|w_j\rangle$'s are in fact orthonormal).

First notice that $I_n \geq |w_i\rangle\langle w_i|$ implies that $kI_n - |w_i\rangle\langle w_i| \geq (k-1)|w_i\rangle\langle w_i|$. Because the $|w_j\rangle$'s are orthonormal it follows that

$$\begin{aligned}
(id_n \otimes \Phi_k)(\rho) &= \sum_{i,j=1}^{k} |v_i\rangle\langle v_j| \otimes (k\langle w_j|w_i\rangle I_n - |w_i\rangle\langle w_j|) \\
&\geq \sum_{i=1}^{k} (k-1)|v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i| - \sum_{\substack{i,j=1 \\ i\neq j}}^{k} |v_i\rangle\langle v_j| \otimes |w_i\rangle\langle w_j| \\
&= \sum_{\substack{i,j=1 \\ i\neq j}}^{k} \Big(|v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i| - |v_i\rangle\langle v_j| \otimes |w_i\rangle\langle w_j| \Big) \\
&= \sum_{i=1}^{k} \sum_{j=i+1}^{k} \Big(|v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i| - |v_i\rangle\langle v_j| \otimes |w_i\rangle\langle w_j| \\
&\quad - |v_j\rangle\langle v_i| \otimes |w_j\rangle\langle w_i| + |v_j\rangle\langle v_j| \otimes |w_j\rangle\langle w_j| \Big) \\
&= \sum_{i=1}^{k} \sum_{j=i+1}^{k} \big(|v_i\rangle \otimes |w_i\rangle - |v_j\rangle \otimes |w_j\rangle \big)\big(\langle v_i| \otimes \langle w_i| - \langle v_j| \otimes \langle w_j| \big) \\
&\geq 0
\end{aligned}$$

∎

We thus see that a necessary condition for $SN(\rho) \leq k$ is that $(id_n \otimes \Phi_k)(\rho) \geq 0$. We can restate this as the following *Generalized Reduction Criterion*:

Theorem 2.3 (Generalized Reduction Criterion)**.** *If $SN(\rho) \leq k$ then $k\rho_2 \otimes I \geq \rho$ and $kI \otimes \rho_1 \geq \rho$.*

As indicated by its name, the Generalized Reduction Criterion generalizes the standard reduction criterion of Horodecki and Horodecki [4]. The following corollary is trivial and thus its proof is omitted.

Corollary 2.4. *If the maximum eigenvalue of $\rho \in M_m \otimes M_n$ is strictly greater than k times the maximum eigenvalue of either of its reduced density matrices, then its Schmidt Number is at least $k+1$.*

2.4. **Relationship with k-Positive Maps.** As was seen in the previous section, $(id_n \otimes \Phi_k)(\rho) \geq 0$ for all $\rho \in M_n \otimes M_n$ with $SN(\rho) \leq k$, where $\Phi_k(\sigma) = k\mathrm{Tr}(\sigma) \cdot I_n - \sigma$. The following theorem shows that this is no coincidence; it occurs precisely because Φ_k is a k-positive map (a fact that is likely well-known to operator algebraists).

From this point on, we will denote the set of k-positive linear maps by $\mathcal{P}_k$ when it is convenient to do so.

Theorem 2.5 (k-Positive Maps)**.** *Let $\Phi : M_n \mapsto M_m$ be a linear map. Then $\Phi \in \mathcal{P}_k$ if and only if*

$$(id_n \otimes \Phi)(\rho) \geq 0 \quad \forall\, \rho \in M_n \otimes M_n \text{ with } SN(\rho) \leq k.$$

The above theorem is well-known in quantum information, and it can be reworked slightly to show that the exact same inequality characterizes the Schmidt Number of density matrices in terms of k-positive maps; a result that was formalized by Terhal and Horodecki [12].

Theorem 2.6 (Terhal and Horodecki [12])**.** *Let $\rho \in M_n \otimes M_n$ be a density matrix. Then $SN(\rho) \leq k$ if and only if*

$$(id_n \otimes \Phi)(\rho) \geq 0 \quad \forall\, \Phi \in \mathcal{P}_k.$$

One special case of this theorem is that ρ is separable if and only if $(id_n \otimes \Phi)(\rho) \geq 0$ for all positive maps Φ.

The original proof of Terhal and Horodecki's Theorem is quite technical, so it will not be proved here. Instead it will come as a corollary of the much more general Theorem 4.5, which will be presented and proved later in this talk.

3. Partially Entanglement Breaking Maps

3.1. **Definition.**

Definition 3.1 (Partially Entanglement Breaking Maps [3])**.** *A completely positive map $\Phi : M_n \mapsto M_m$ is said to be k-Partially Entanglement Breaking (k-PEB) if $SN((id_n \otimes \Phi)(\rho)) \leq k$ for all $0 \leq \rho \in M_n \otimes M_n$. The set of k-PEB maps will be denoted by $\mathcal{B}_k$.*

As with the Schmidt Number for density matrices, there are two special cases to note here:

- Entanglement Breaking Maps [5] are exactly the 1-PEB maps.
- The set of $\min\{m, n\}$-PEB maps contains all completely positive maps.

3.2. **Characterization.** One fact that will be useful from time to time is that $\Phi \in \mathcal{B}_k$ if and only if it can be written in the following form (which will be proved as part of the upcoming characterization).

$$\Phi(\rho) = \sum_p \sum_{q,r=1}^{k} \langle v_{p,q}|\rho|v_{p,r}\rangle |w_{p,q}\rangle\langle w_{p,r}| \tag{2}$$

In the following theorem, as well as the rest of this talk, $|e\rangle$ will denote the maximally entangled state $|e\rangle = \sum_{j=1}^{n} |e_j\rangle \otimes |e_j\rangle$, where $\{|e_j\rangle\}$ is the standard orthonormal basis of $\mathbb{C}^n$. Thus, $(id_n \otimes \Phi)(|e\rangle\langle e|)$ is the Choi matrix [2] (which we will often denote by C_Φ) of Φ.

We could equivalently replace $|e\rangle$ with any other maximally-entangled pure state for the duration of this talk.

Theorem 3.2 (PEB Characterization)**.** *Let $\Phi : M_n \mapsto M_m$ be a completely positive linear map. Then the following are equivalent:*

a) $\Phi \in \mathcal{B}_k$.
b) $(id_n \otimes \Phi)(|e\rangle\langle e|)$ is a state with Schmidt number at most k.
c) Φ can be written in the form (2) from earlier.
d) Φ can be written in operator sum form using Kraus operators of rank at most k.
e) $\Psi \circ \Phi$ is completely positive for all k-positive maps Ψ.
f) $\Phi \circ \Psi$ is completely positive for all k-positive maps Ψ.

Proof. The implication a) $\Rightarrow$ b) is trivial. To see b) $\Rightarrow$ c) note that

$$(id_n \otimes \Phi)(|e\rangle\langle e|) = \sum_{i,j} |e_i\rangle\langle e_j| \otimes \Phi(|e_i\rangle\langle e_j|),$$

and this state must have Schmidt Number at most k. Thus there must exist vectors $\{|\psi_p\rangle\}$ so that $|\psi_p\rangle = \sum_{q=1}^{k} |v_{p,q}\rangle \otimes |w_{p,q}\rangle$ and

$$\begin{aligned}\sum_{i,j} |e_i\rangle\langle e_j| \otimes \Phi(|e_i\rangle\langle e_j|) &= \sum_p |\psi_p\rangle\langle\psi_p| \\ &= \sum_p \sum_{q,r=1}^{k} |v_{p,q}\rangle\langle v_{p,r}| \otimes |w_{p,q}\rangle\langle w_{p,r}|.\end{aligned}$$

Now let Ω be a map of the form (2) defined by

$$\Omega(\rho) = \sum_p \sum_{q,r=1}^{k} \overline{\langle v_{p,q}|}\rho\overline{|v_{p,r}\rangle} |w_{p,q}\rangle\langle w_{p,r}|,$$

where $\overline{|v_{p,r}\rangle} = \sum_j \langle v_{p,r}|e_j\rangle|e_j\rangle$. Then by recalling that $|v_{p,r}\rangle = \sum_j \langle e_j|v_{p,r}\rangle|e_j\rangle$ it is not difficult to verify that

$$\begin{aligned}(id_n \otimes \Omega)(|e\rangle\langle e|) &= \sum_{i,j,p}\sum_{q,r=1}^{k} |e_i\rangle\langle e_j| \otimes \overline{\langle v_{p,q}|e_i\rangle\langle e_j|v_{p,r}\rangle}|w_{p,q}\rangle\langle w_{p,r}| \\ &= \sum_{p}\sum_{q,r=1}^{k} |v_{p,q}\rangle\langle v_{p,r}| \otimes |w_{p,q}\rangle\langle w_{p,r}|.\end{aligned}$$

Thus $(id_n \otimes \Phi)(|e\rangle\langle e|) = (id_n \otimes \Omega)(|e\rangle\langle e|)$ and so it follows that $\Phi = \Omega$ because linear maps are determined by their action on the orthonormal basis $|e_i\rangle\langle e_j|$.

To see c) $\Rightarrow$ d) simply note that

$$\begin{aligned}\Phi(\rho) &= \sum_{p}\sum_{q,r=1}^{k} \langle v_{p,q}|\rho|v_{p,r}\rangle|w_{p,q}\rangle\langle w_{p,r}| \\ &= \sum_{p=1}\Big[\sum_{q=1}^{k} |w_{p,q}\rangle\langle v_{p,q}|\Big]\,\rho\,\Big[\sum_{r=1}^{k} |v_{p,r}\rangle\langle w_{p,r}|\Big],\end{aligned}$$

and each $\big[\sum_{q=1}^{k} |w_{p,q}\rangle\langle v_{p,q}|\big]$ has rank at most k.

To see d) $\Rightarrow$ a), note that we can write a rank 1 density matrix $\rho \in M_n \otimes M_n$ as $\rho = |x\rangle\langle x|$, where $|x\rangle = \sum_i |y_i\rangle \otimes |z_i\rangle$. Thus,

$$\begin{aligned}(id_n \otimes \Phi)(\rho) &= \sum_{i,j} |y_i\rangle\langle y_j| \otimes \sum_{p}\sum_{q,r=1}^{k} |w_{p,q}\rangle\langle v_{p,q}|z_i\rangle\langle z_j|v_{p,r}\rangle\langle w_{p,r}| \\ &= \sum_{i,j,p}\sum_{q,r=1}^{k} \langle v_{p,q}|z_i\rangle\langle z_j|v_{p,r}\rangle\, |y_i\rangle\langle y_j| \otimes |w_{p,q}\rangle\langle w_{p,r}| \\ &= \sum_{p}\Big[\sum_{q=1}^{k} |x_{p,q}\rangle \otimes |w_{p,q}\rangle\Big]\Big[\sum_{r=1}^{k}\langle x_{p,r}| \otimes \langle w_{p,r}|\Big],\end{aligned}$$

where $|x_{p,q}\rangle = \sum_i \langle v_{p,q}|z_i\rangle|y_i\rangle$. The above state has Schmidt Number at most k, so the implication follows from linearity.

We have established the equivalence of conditions a) through d).

The equivalence of a) and e) comes as an immediate corollary of Terhal and Horodecki's Theorem from earlier.

The equivalence of e) and f) comes from the following three simple facts: Ψ is k-positive if and only if $\Psi^\dagger$ is k-positive, Φ is k-PEB if and only if $\Phi^\dagger$ is k-PEB (which can be seen as a corollary of the equivalence of a) and d)), and $(\Psi \circ \Phi)^\dagger = \Phi^\dagger \circ \Psi^\dagger$. ■

The various equivalences of the above characterization were each proved in [3] and can be regarded as a generalization of the characterization of Entanglement Breaking Maps of [5]. The proof above is presented for completeness because it is elementary, and because it establishes the equivalence of a) through d) in a constructive manner.

3.3. **Results.** The following theorem generalizes a result of Horodecki, Shor and Ruskai [5] that says that Entanglement Breaking Channels can't be represented with fewer than n Kraus operators.

Theorem 3.3. *If $k \geq 1$ and $\Phi : M_n \mapsto M_m$ is a quantum channel (that is, a trace-preserving completely positive linear map) that can be written with fewer than $\lceil n/k \rceil$ Kraus operators, then Φ is not k-PEB.*

Proof. Suppose that the fewest Kraus operators that Φ can be written with is r. Then $r = rank(C_\Phi)$ by a result of Choi [2].

We also know that $Tr_2(C_\Phi) = I_n$ since Φ is trace-preserving. Since C_Φ has rank $r < \frac{n}{k}$ and has trace equal to n, it follows that $\lambda_{max} > k$, where λ_{max} is the largest eigenvalue of C_Φ. If follows from the Generalized Reduction Criterion that $SN(C_\Phi) > k$ and so the characterization theorem tells us that Φ is not k-PEB. ∎

4. Cones of Positive Maps

4.1. **Jamiolkowski Isomorphism.** The Jamiolkowski Isomorphism [7] is a way of associating with every $\Phi \in \mathcal{L}(M_n, M_m)$ a unique $C_\Phi \in M_{nm} \cong M_n(M_m)$. As the notation suggests, the isomorphism indeed just takes Φ to its Choi matrix, $(id_n \otimes \Phi)(|e\rangle\langle e|)$.

It is easy to see that this is really an isomorphism because $(id_n \otimes \Phi)(|e\rangle\langle e|)$ is just an $n \times n$ block matrix of the action of Φ on each of the standard matrix units $|e_i\rangle\langle e_j|$.

4.2. **Cones and Dual Cones.** Notice that if $\lambda \in \mathbb{R}_+$ and $\Phi \in \mathcal{CP}$ then $\lambda \cdot \Phi \in \mathcal{CP}$. Thus, the set $C_{\mathcal{CP}}$ of Choi matrices of completely positive maps forms a cone in M_{nm} (indeed, $C_{\mathcal{CP}} = M_{nm}^+$).

Similarly, it is easy to verify that the sets $C_{\mathcal{P}_k}$ and $C_{\mathcal{B}_k}$ are also cones in M_{nm}. These two cones are the primary motivation for the definition and investigation of *right-CP invariant cones* in the next section.

Given a cone $\mathcal{C} \subseteq M_{nm}^{sa}$ of self-adjoint matrices, one can define its *dual cone* in the usual way:

$$\mathcal{C}^d \equiv \big\{ D \in \mathcal{C} \,|\, \langle C|D\rangle_{HS} \equiv \mathrm{Tr}(CD) \geq 0 \ \ \forall\, C \in \mathcal{C} \big\}.$$

Given a cone $\mathcal{S} \subseteq \mathcal{L}(M_n, M_m)$, we may use the Choi-Jamiolkowski Isomorphism to find the cone $C_{\mathcal{S}} \subseteq M_{nm}$, use the dual cone definition above to find $C_{\mathcal{S}}^d$, and use the Choi-Jamiolkowski Isomorphism again to find another cone of linear maps, which we will denote by $\mathcal{S}^D$:

$$\mathcal{S}^D \equiv \big\{ \Phi \in \mathcal{L}(M_n, M_m) \,|\, C_\Phi \in C_{\mathcal{S}}^d \big\}.$$

We will call $\mathcal{S}^D$ the *dual cone* of $\mathcal{S}$. It will be clear from the context whether we are talking about the dual cone of matrices or the dual cone of linear maps, though this distinction is not usually very important due to the Jamiolkowski Isomorphism.

4.3. **The Dual Cone of k-Positive Maps.** In the $k=1$ case, it is known [11] that $\Phi \in \mathcal{P}_1$ if and only if $(\langle x| \otimes \langle y|)C_\Phi(|x\rangle \otimes |y\rangle) \geq 0$ for all $|x\rangle \in \mathbb{C}^n, |y\rangle \in \mathbb{C}^m$.

The above fact can be used to show that $\mathcal{P}_1^D = \mathcal{B}_1$ and $\mathcal{B}_1^D = \mathcal{P}_1$. That is, the Entanglement Breaking Maps are the dual cone of the Positive Maps, and vice-versa.

The same result holds for higher k, as has been hinted at by the recent literature on the subject [8, 10].

It was proved in [8] that Φ is k-positive if and only if $\langle v|C_\Phi|v\rangle \geq 0$ for all vectors $|v\rangle$ with Schmidt Rank at most k – this fact can also be seen as a corollary of Terhal and Horodecki's theorem from earlier. We will prove it here for completeness.

Theorem 4.1. *$\Phi \in \mathcal{P}_k$ if and only if $\langle v|C_\Phi|v\rangle \geq 0$ for all vectors $|v\rangle \in \mathbb{C}^{nm}$ with Schmidt Rank at most k.*

Proof. The proof will simply make use of the fact that $(id_k \otimes \Phi)$ is positive if and only if $(\langle x| \otimes \langle y|)C_{id_k \otimes \Phi}(|x\rangle \otimes |y\rangle) \geq 0$ for all $|x\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n, |y\rangle \in \mathbb{C}^k \otimes \mathbb{C}^m$.

First write $|x\rangle = \sum_{i=1}^k |e_i\rangle \otimes |x_i\rangle$ and $|y\rangle = \sum_{i=1}^k |e_i\rangle \otimes |y_i\rangle$. Then

$$|x\rangle \otimes |y\rangle = \sum_{i,j=1}^k |e_i\rangle \otimes |x_i\rangle \otimes |e_j\rangle \otimes |y_j\rangle$$

and after some simplification we see that

$$\begin{aligned}
\Phi \in \mathcal{P}_k &\iff (\langle x| \otimes \langle y|)C_{id_k \otimes \Phi}(|x\rangle \otimes |y\rangle) \geq 0 \quad \forall |x\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n, |y\rangle \in \mathbb{C}^k \otimes \mathbb{C}^m \\
&\iff \sum_{i,l=1}^k \sum_{r,s=1}^n (\langle x_l| \otimes \langle y_l|)|e_r\rangle\langle e_s| \otimes \Phi(|e_r\rangle\langle e_s|)(|x_i\rangle \otimes |y_j\rangle) \geq 0 \quad \forall \{|x_l\rangle\}, \{|y_l\rangle\} \\
&\iff \sum_{i=1}^k \Big[\langle x_i| \otimes \langle y_i|\Big] C_\Phi \sum_{i=1}^k \Big[|x_i\rangle \otimes |y_i\rangle\Big] \geq 0 \quad \forall \{|x_i\rangle\} \in \mathbb{C}^n, \{|y_i\rangle\} \in \mathbb{C}^m \\
&\iff \langle v|C_\Phi|v\rangle \geq 0 \quad \forall |v\rangle \text{ with Schmidt Rank at most } k.
\end{aligned}$$

∎

Similarly, we saw earlier (Theorem 3.2) that $\Phi \in \mathcal{B}_k$ if and only if $C_\Phi = \sum_j |v_j\rangle\langle v_j|$, where each $|v_j\rangle$ has Schmidt rank at most k.

Putting these results together, we thus find that the dual cone of the k-PEB maps are exactly the k-positive maps:

$$\begin{aligned}
&\Phi \in \mathcal{B}_k^D \\
&\iff \mathrm{Tr}(C_\Phi \rho) \geq 0 \quad \forall\, \rho \in C_{\mathcal{B}_k} \\
&\iff \mathrm{Tr}(C_\Phi |v\rangle\langle v|) \geq 0 \quad \forall\, |v\rangle \text{ with Schmidt rank at most } k \\
&\iff \langle v|C_\Phi|v\rangle \geq 0 \quad \forall\, |v\rangle \text{ with Schmidt rank at most } k \\
&\iff \Phi \in \mathcal{P}_k.
\end{aligned}$$

Indeed, because $C_{\mathcal{B}_k}$ is a closed, convex cone it follows that $\mathcal{P}_k^D = \mathcal{B}_k$ as well.

4.4. **Other Cones of Positive Maps.** We have seen several ways of characterizing positive linear maps $\Phi : M_n \mapsto M_m$ by looking at cones on which $(id_n \otimes \Phi)$ is positive. Here we will see how these results extend to cones of positive maps other than PEB or k-positive maps.

Definition 4.2. *Let $\mathcal{S} \subseteq \mathcal{L}(M_n, M_m)$ be a set of positive linear maps. $\mathcal{S}$ will be said to be a* right CP-invariant cone *if $\Phi \circ \Psi \in \mathcal{S}$ whenever $\Phi \in \mathcal{S}$ and $\Psi \in \mathcal{CP}$.*

- The definition of a *left CP-invariant cone* should be obvious.
- It is clear that $\mathcal{P}_k$ is both left and right CP-invariant.
- It is clear that $\mathcal{B}_k$ is right CP-invariant. The fact that it is left CP-invariant comes from Theorem 2.2.
- In [9] a *mapping cone* was defined to be a cone that is both *left* and *right CP-invariant*, though I will not use this term in this talk.

Before delving too deep into results concerning right CP-invariant cones, it will be useful to formally present some useful (albeit trivial) properties.

Lemma 4.3. *$\mathcal{S}$ is a right CP-invariant cone if and only if $\mathcal{S}^\dagger \equiv \{\Phi^\dagger | \Phi \in \mathcal{S}\}$ is a left CP-invariant cone.*

Proof. Simply notice that $\Psi \in \mathcal{CP} \Leftrightarrow \Psi^\dagger \in \mathcal{CP}$ and $(\Phi \circ \Psi)^\dagger = \Psi^\dagger \circ \Phi^\dagger$. ■

The above lemma shows that all of the following results about right CP-invariant cones can easily be modified to be stated in terms of left CP-invariant cones.

Lemma 4.4. *Let $\mathcal{S}$ be a right CP-invariant cone. Then the following are equivalent:*
a) $\Phi \in \mathcal{S}$
b) $(id_n \otimes \Phi)(\rho) \in C_{\mathcal{S}}$ for all $0 \leq \rho \in M_n \otimes M_n$

Furthermore, if a set $\mathcal{S}' \subseteq \mathcal{L}(M_n, M_m)$ is such that a) $\Rightarrow$ b), then $\mathcal{S}'$ must be a right CP-invariant cone.

Proof. The implication b) $\Rightarrow$ a) is trivial because we can take $\rho = |e\rangle\langle e|$.

To see a) $\Rightarrow$ b), note that because $\mathcal{S}$ is a regular cone, we have that $\Phi \circ \Psi \in \mathcal{S}$ for all $\Psi \in \mathcal{CP}$. Thus $(id_n \otimes \Phi \circ \Psi)(|e\rangle\langle e|) = (id_n \otimes \Phi)(C_\Psi) \in C_{\mathcal{S}}$ for all $\Psi \in \mathcal{CP}$. The result then comes from Choi's result [2] that $\Psi \in \mathcal{CP} \Leftrightarrow C_\Psi \geq 0$ and the Jamiolkowski Isomorphism.

The final claim comes using the exact same reasoning used to establish the implication a) $\Rightarrow$ b) in reverse. ■

In terms of k-PEB maps, the above lemma gives us exactly the equivalence of a) and b) in the PEB Characterization Theorem.

Theorem 4.5. *Let $\mathcal{S} \subseteq L(M_n, M_m)$ be a cone, and let $\mathcal{C} \subseteq \mathcal{S}$. Consider the following four properties:*
a) $\rho \in C_{\mathcal{S}}^d$
b) $\langle e|(id_n \otimes \Phi^\dagger)(\rho)|e\rangle \geq 0 \quad \forall\, \Phi \in \mathcal{S}$
c) $(id_n \otimes \Phi^\dagger)(\rho) \geq 0 \quad \forall\, \Phi \in \mathcal{S}$
d) $(id_n \otimes \Phi^\dagger)(\rho) \geq 0 \quad \forall\, \Phi \in \mathcal{C}$.

Then

(1) a) $\Leftrightarrow$ b),
(2) If $\mathcal{S}$ is right CP-invariant then a) and c) are equivalent, and
(3) If $\mathcal{S}$ is right CP-invariant and $\mathcal{C} \circ \mathcal{CP} = \mathcal{S}$ then a) and d) are equivalent.

Proof. To see (1), note that

$$\begin{aligned} & \langle e|(id_n \otimes \Phi^\dagger)(\rho)|e\rangle \geq 0 \quad \forall\, \Phi \in \mathcal{S} \\ \Longleftrightarrow\; & \mathrm{Tr}(|e\rangle\langle e|(id_n \otimes \Phi)^\dagger(\rho)) \geq 0 \quad \forall\, \Phi \in \mathcal{S} \\ \Longleftrightarrow\; & \mathrm{Tr}((id_n \otimes \Phi)(|e\rangle\langle e|)\rho) \geq 0 \quad \forall\, \Phi \in \mathcal{S} \\ \Longleftrightarrow\; & \mathrm{Tr}(C_\Phi \rho) \geq 0 \quad \forall\, C_\Phi \in C_{\mathcal{S}} \\ \Longleftrightarrow\; & \rho \in C_{\mathcal{S}}^d. \end{aligned}$$

The implication c) $\Rightarrow$ b) is trivial. To see the reverse implication, note that if $\mathcal{S}$ is a right CP-invariant cone, we can replace $\Phi^\dagger$ with $\Psi^\dagger \circ \Phi^\dagger$ where $\Psi \in \mathcal{CP}$ is arbitrary. We thus have that

$$\begin{aligned} & \langle e|(id_n \otimes \Psi^\dagger \circ \Phi^\dagger)(\rho)|e\rangle \geq 0 \quad \forall\, \Phi \in \mathcal{S}, \Psi \in \mathcal{CP} \\ \Longrightarrow\; & \Big(\sum_{j=1}^n \langle e_j| \otimes \langle e_j|X\Big)(id_n \otimes \Phi^\dagger)(\rho)\Big(\sum_{j=1}^n |e_j\rangle \otimes X^\dagger\langle e_j|\Big) \geq 0 \quad \forall\, \Phi \in \mathcal{S},\, \forall X \in M_{n,m} \\ \Longrightarrow\; & \Big(\sum_{j=1}^n \langle e_j| \otimes \langle x_j|\Big)(id_n \otimes \Phi^\dagger)(\rho)\Big(\sum_{j=1}^n |e_j\rangle \otimes |x_j\rangle\Big) \geq 0 \quad \forall\, \Phi \in \mathcal{S},\, \forall X \in M_{n,m}, \end{aligned}$$

where $\langle x_j|$ is the j^{th} row of X. Since any $|x\rangle$ can be written in the form $|x\rangle = \sum_{j=1}^n |e_j\rangle \otimes |x_j\rangle$, (2) established.

Now note that c) $\Rightarrow$ d) is trivial. To see that a) $\Leftrightarrow$ d) in the case of (3), we will show that d) $\Rightarrow$ b). To this end, check that

$$\begin{aligned} & (id_n \otimes \Phi^\dagger)(\rho) \geq 0 \quad \forall\, \Phi \in \mathcal{C} \\ \Longrightarrow\; & \langle \psi|(id_n \otimes \Phi^\dagger)(\rho)|\psi\rangle \geq 0 \quad \forall\, \Phi \in \mathcal{C}, |\psi\rangle \in \mathbb{C}^{n^2} \\ \Longrightarrow\; & \mathrm{Tr}(C_\Psi(id_n \otimes \Phi^\dagger)(\rho)) \geq 0 \quad \forall\, \Phi \in \mathcal{C}, 0 \leq C_\Psi \in M_{n^2} \\ \Longrightarrow\; & \mathrm{Tr}((id_n \otimes \Psi)(|e\rangle\langle e|)(id_n \otimes \Phi^\dagger)(\rho)) \geq 0 \quad \forall\, \Phi \in \mathcal{C}, \Psi \in \mathcal{CP} \\ \Longrightarrow\; & \mathrm{Tr}((id_n \otimes \Phi \circ \Psi)(|e\rangle\langle e|)\rho) \geq 0 \quad \forall\, \Phi \in \mathcal{C}, \Psi \in \mathcal{CP} \\ \Longrightarrow\; & \mathrm{Tr}((id_n \otimes \Phi')(|e\rangle\langle e|)\rho) \geq 0 \quad \forall\, \Phi' \in \mathcal{S} \\ \Longrightarrow\; & \langle e|(id_n \otimes {\Phi'}^\dagger)(\rho)|e\rangle \geq 0 \quad \forall\, \Phi' \in \mathcal{S}, \end{aligned}$$

where the fact that every $\rho \geq 0$ can be associated with a completely positive map was used in the third line. ∎

If we take $\mathcal{S} = \mathcal{P}_k$ then the equivalence of a) and c) in the above theorem gives us exactly Terhal and Horodecki's Theorem.

Lemma 4.6. *Let $\Phi, \Psi \in \mathcal{L}(M_n, M_m)$. Then $(id_n \otimes \Psi^\dagger)(C_\Phi) \geq 0 \Leftrightarrow (id_n \otimes \Phi^\dagger)(C_\Psi) \geq 0$.*

Proof.

$$\begin{aligned}&(id_n \otimes \Psi^\dagger)(C_\Phi) \geq 0\\ \Longleftrightarrow\ &(id_n \otimes \Psi^\dagger \circ \Phi)(|e\rangle\langle e|) \geq 0\\ \Longleftrightarrow\ &\Psi^\dagger \circ \Phi \in \mathcal{CP}\\ \Longleftrightarrow\ &\Phi^\dagger \circ \Psi \in \mathcal{CP}\\ \Longleftrightarrow\ &(id_n \otimes \Phi^\dagger \circ \Psi)(|e\rangle\langle e|) \geq 0\\ \Longleftrightarrow\ &(id_n \otimes \Phi^\dagger)(C_\Psi) \geq 0\end{aligned}$$

■

Corollary 4.7. *Let $\mathcal{S} \subseteq \mathcal{L}(M_n, M_m)$ be a cone, and let $\mathcal{C} \subseteq \mathcal{S}$. Consider the following four properties:*
a) $\Psi \in \mathcal{S}^D$
b) $\langle e|(id_n \otimes \Psi^\dagger)(\rho)|e\rangle \geq 0 \quad \forall\, \rho \in C_{\mathcal{S}}$
c) $(id_n \otimes \Psi^\dagger)(\rho) \geq 0 \quad \forall\, \rho \in C_{\mathcal{S}}$
d) $(id_n \otimes \Psi^\dagger)(\rho) \geq 0 \quad \forall\, \rho \in C_{\mathcal{C}}$

Then
(1) a) $\Leftrightarrow$ b),
(2) If $\mathcal{S}$ is right CP-invariant then a) and c) are equivalent, and
(3) If $\mathcal{S}$ is right CP-invariant and $\mathcal{C} \circ \mathcal{CP} = \mathcal{S}$ then a) and d) are equivalent.

Proof. (1) is proved in the same way as Theorem 4.5. (2) and (3) follow from the analogous results in Theorem 4.5 via Lemma 4.6. ■

If we take $\mathcal{S} = \mathcal{B}_k$ then the equivalence of a) and c) in the above corollary gives us exactly Theorem 2.5.

Corollary 4.8. *Let $\mathcal{S} \subseteq \mathcal{L}(M_n, M_m)$ be a right CP-invariant cone. Then the following are equivalent:*
a) $\Psi \in \mathcal{S}^D$
b) $\Phi^\dagger \circ \Psi \in \mathcal{CP}$ *for all* $\Phi \in \mathcal{S}$

Proof. The result comes from associating $\rho \in C_{\mathcal{S}}^d$ with a map $\Psi \in \mathcal{S}^D$ in Theorem 4.5. ■

By taking $\mathcal{S} = \mathcal{P}_k$ in the above corollary, we get the equivalence of a), e) and f) in the PEB Characterization Theorem.

5. Conclusions

We have built up from the basic ideas of Schmidt Rank and Schmidt Number to examine Partially Entanglement Breaking Maps. We have seen that many of the properties of Partially Entanglement Breaking maps follow precisely because they form a right CP-invariant cone, and their interplay with the k-positive maps follows because they are dual cones of each other.

Many natural cones of quantum maps are not right CP-invariant – trace-preserving maps, unital maps, randomized unitary channels, and so on. Thus a natural next step is to extend the results of Section 4.4, perhaps to *right unitarily-invariant cones*, and explore the implications that follow.

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