

COMPUTING STABILIZED NORMS FOR QUANTUM OPERATIONS VIA THE THEORY OF COMPLETELY BOUNDED MAPS

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ABSTRACT. The diamond and completely bounded norms for linear maps play an increasingly important role in quantum information science, providing fundamental stabilized distance measures for differences of quantum operations. Based on the theory of completely bounded maps, we formulate an algorithm to compute the norm of an arbitrary linear map. We present an implementation of the algorithm *via* Maple, discuss its efficiency, and consider the case of differences of unitary maps.

1. INTRODUCTION

The need for physically significant and computable distance measures for quantum operations and channels is of fundamental importance in quantum information science [20]. Most importantly, it is often necessary to determine how far apart two quantum operations, represented by completely positive maps, are from each other in some meaningful sense. The *diamond norm* was introduced in [16] for this purpose. It arises from physical considerations and satisfies the important stability property desired for such measures [1, 11]. Interestingly, the diamond norm is intimately related to the *norm of complete boundedness*, a notion that has been studied in operator theory for different reasons over the past four decades [21]. On finite dimensional Hilbert space, every linear map has a finite completely bounded (CB) norm. Thus, CB maps are precisely the linear maps in the finite dimensional case. In operator theory, CB maps are the natural maps between certain objects called operator spaces. Computing the norms of CB maps between certain operator spaces introduced in [26] has provided the impetus for recent progress on multiplicativity conjectures for quantum channels [9, 10, 12, 31]. CB maps and norms have also arisen in a wide variety of other recent investigations in quantum information science, including [14, 15, 18, 19, 25, 27, 30], though the CB terminology has not always been used.

In this paper, based on the classical and contemporary theory of completely bounded maps, we formulate an algorithm to compute completely bounded and diamond norms for arbitrary linear maps on finite dimensional Hilbert space. Along the way, we also provide a brief introduction to completely bounded map theory, including the generalized Stinespring theorem and Choi-Kraus representation for such maps. We then present an

implementation of the algorithm *via* Maple. We also discuss the algorithm's efficiency, and note how it is potentially optimal. Our approach to computing these norms is distinct from other known approaches, such as the use of semidefinite programming [29].

In the next section we recall basic properties of the diamond and completely bounded norm, showing how to interpolate between the two. This is followed by the introduction to completely bounded maps; our presentation here is motivated by that of [21]. In the penultimate section we describe the theoretical formulation of the algorithm, and apply it to derive a geometric formula for the case of differences of unitary maps. In the final section we exhibit code for the Maple implementation of the algorithm, giving an explanation of each subroutine.

2. LINEAR MAPS AND STABILIZED NORMS

We shall write M_n for the set of $n \times n$ complex matrices, and regard it as the set of operators acting on an n -dimensional Hilbert space represented as matrices in a given orthonormal basis. *Quantum operations* or *channels* are represented by linear maps $\phi : M_n \rightarrow M_k$ that are completely positive and trace preserving (in the Schrodinger picture) or unital (in the Heisenberg picture). The dual map $\phi^\dagger : M_k \rightarrow M_n$ is defined via the Hilbert-Schmidt inner product $\text{Tr}(\phi(A)B) = \text{Tr}(A\phi^\dagger(B))$.

In quantum information one is often interested in properties of differences $\phi - \psi$ between pairs of quantum operations. Such a difference is still a linear map (a ‘‘superoperator’’), though not necessarily completely positive. In fact, every linear map can be decomposed as a linear combination of at most four completely positive maps. This parallels the corresponding statement about general operators and positive operators on Hilbert space, though the proof is more delicate [32, 21].

The 1-norm of a linear map ϕ is given by $\|\phi\|_1 = \sup_{\|X\|_1 \leq 1} \|\phi(X)\|_1$, where $\|X\|_1 = \text{Tr}|X|$. The operator norm of ϕ is $\|\phi\| = \sup_{\|X\| \leq 1} \|\phi(X)\|$, where $\|X\| = \sup_{\|\xi\| \leq 1} \|X\xi\|$, and ξ ranges over the unit ball of the domain Hilbert space for X . Every norm $\|\phi\|$ defines a distance measure $d(\phi, \psi) = \|\phi - \psi\|$. Neither of the distance measures defined by these norms satisfies the *stabilization property* for distance measures of superoperators [16, 1, 11]:

$$d(id_m \otimes \phi, id_m \otimes \psi) = d(\phi, \psi) \quad \forall m \geq 1.$$

This property implies that the distance between quantum operations is unaffected by any ancillary quantum system that is independent of the original system.

The diamond norm is defined in [16, 1] through partial traces, but it is shown to be equivalent to the following quantity.

Definition 1. *For a linear map $\phi : M_n \rightarrow M_k$, define*

$$\|\phi\|_\diamond = \|id_n \otimes \phi\|_1.$$

Though not obvious, the stabilization property is satisfied by $\|\cdot\|_\diamond$. One way to see this is through a connection with the completely bounded norm, to which we now turn.

Definition 2. For a linear map $\phi : M_n \rightarrow M_k$, define

$$\|\phi\|_{cb} = \sup_{m \geq 1} \|id_m \otimes \phi\|,$$

which we call the **completely bounded norm** or the **cb-norm**.

As a convenience, we adopt the notation $\phi_m \equiv id_m \otimes \phi$. It is easily checked that $\|\phi_m\| \leq \|\phi_{m+1}\|$ and $\|\phi_m\| \leq m\|\phi\|$ [21, Chapter 1]. Note also that $\|\mathcal{U}\phi\|_{cb} = \|\phi\|_{cb} = \|\phi\mathcal{U}\|_{cb}$ for every unitarily implemented map $\mathcal{U}(\cdot) = U(\cdot)U^\dagger$. Furthermore, we have $\|\phi_1 \otimes \phi_2\|_{cb} = \|\phi_1\|_{cb}\|\phi_2\|_{cb}$, and $\|id\|_{cb} = 1$. The identification in Theorem 3 shows the corresponding properties hold for the diamond norm.

It is possible to relate the completely bounded norm and the diamond norm as follows. We first note that by a theorem of Smith [28, 21] the cb-norm stabilizes in the sense that for a map $\phi : M_n \rightarrow M_k$ we have that $\|\phi\|_{cb} = \|id_k \otimes \phi\|$. Then we make use of the duality relationship [4] given by $\|\phi\| = \sup_{\|X\|_1 \leq 1} \|\phi^\dagger(X)\|_1$, to obtain

$$\|\phi\|_{cb} = \|id_k \otimes \phi\| = \sup_{\|X\|_1 \leq 1} \|(id_k \otimes \phi^\dagger)(X)\|_1 = \|\phi^\dagger\|_\diamond,$$

since $\phi^\dagger : M_k \rightarrow M_n$.

Thus, we have the following, which also includes an upper bound [28].

Theorem 3. Let $\phi : M_n \rightarrow M_k$, be a linear map, then

$$\|\phi\|_{cb} = \|\phi^\dagger\|_\diamond = \|\phi_k\| \leq k\|\phi\|.$$

In summary, using the fact that these maps appear as dual pairs, we see that for $\psi : M_m \rightarrow M_j$, $\|\psi\|_\diamond = \|\psi^\dagger\|_{cb} \leq m\|\psi\|$ and that the stability of the diamond norm [16, 1] is the dual version of Smith's stability for the cb-norm [28]. A more refined general upper bound is discussed in the next section.

3. COMPLETELY BOUNDED MAP PRIMER

We next give a compressed introduction to completely bounded maps on arbitrary operator spaces. As noted previously, in the finite dimensional case CB maps on M_n are precisely the linear maps $\phi : M_n \rightarrow M_k$. However, the important structural results reviewed in this section are best viewed in the more general setting of CB maps as in [21]. In any event, if the reader wishes to move directly to the algorithm, this section can be skipped save for the structural result Theorem 19 and the norm description of Corollary 20.

Given a (separable) Hilbert space \mathcal{H} , we denote the set of (bounded) linear operators on \mathcal{H} by $B(\mathcal{H})$. Given operators, $T_{i,j} \in B(\mathcal{H}), 1 \leq i \leq m, 1 \leq j \leq n$, we identify the $m \times n$ matrix of operators, $(T_{i,j})$, with an operator from $\mathcal{H}^{(n)} = \mathcal{H} \oplus \dots \oplus \mathcal{H}$ (n copies) to $\mathcal{H}^{(m)} = \mathcal{H} \oplus \dots \oplus \mathcal{H}$ (m

copies) by regarding vectors in these spaces as columns and performing matrix multiplication. That is, we identify $M_{m,n}(B(\mathcal{H})) \equiv B(\mathcal{H}^{(n)}, \mathcal{H}^{(m)})$. This endows $M_{m,n}(B(\mathcal{H}))$ with a norm and this collection of norms on $B(\mathcal{H})$ is often referred to as the set of *matrix norms* on $B(\mathcal{H})$.

Definition 4. *Let \mathcal{H} be a Hilbert space and let $\mathcal{M} \subseteq B(\mathcal{H})$ be a subspace. Then the inclusion, $M_{m,n}(\mathcal{M}) \subseteq M_{m,n}(B(\mathcal{H}))$ endows this vector space with a collection of matrix norms and we call, \mathcal{M} , together with this collection of matrix norms on $M_{m,n}(\mathcal{M})$ a (concrete) **operator space**. When $m = n$, we set $M_n(\mathcal{M}) = M_{n,n}(\mathcal{M})$.*

Thus, an operator space carries not just an inherited norm structure, but these additional matrix norms.

For basic properties of C*-algebras we point the reader to [8]. C*-algebras are defined abstractly, but every abstract algebra is isomorphic to a concrete C*-algebra given by a subalgebra of some $B(\mathcal{H})$ that is closed under both the operator norm ($\|\cdot\|$) and adjoint (\dagger) operation. If \mathcal{A} is any C*-algebra and $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$ is a one-to-one \dagger -homomorphism (and hence an isometry), then the collection of norms on $M_{m,n}(\pi(\mathcal{A}))$ is independent of the particular representation π , and hence, the operator space structure of a C*-algebra is independent of the particular (faithful) representation. Hence, each subspace $\mathcal{M} \subseteq \mathcal{A}$ is also endowed with a particular collection of matrix norms and so we also refer to a subspace of a C*-algebra as an operator space, when we wish to emphasize its matrix norm structure. We now give the definition of a completely bounded map in the general operator space setting.

Definition 5. *Given a C*-algebra \mathcal{A} , an operator space $\mathcal{M} \subseteq \mathcal{A}$, and a linear map, $\phi : \mathcal{M} \rightarrow B(\mathcal{H})$, we define $\phi_n : M_n(\mathcal{M}) \rightarrow M_n(B(\mathcal{H}))$ by $\phi_n((a_{i,j})) = (\phi(a_{i,j}))$. We call ϕ **completely bounded**, if*

$$\|\phi\|_{cb} \equiv \sup_n \|\phi_n\|,$$

is finite. Here $\|\phi_n\| = \sup \{\|\phi_n(A)\| : A \in M_n(\mathcal{M}), \|A\| \leq 1\}$.

More generally, any time that \mathcal{M} and \mathcal{N} are two spaces, both endowed with a family of matrix norms, then one can define the completely bounded norm of a map $\phi : \mathcal{M} \rightarrow \mathcal{N}$ in analogy with the above definition.

Recalling the upper bound of Theorem 3, for maps whose domain is M_n and range an arbitrary operator space, a result of Haagerup shows that, in general, $\|\psi\|_{cb} \neq \|\psi_m\|$, no matter how large one takes m [21, p. 114], but we do have an upper bound. This result is not explicitly in the literature so we provide a proof below, that uses some concepts we will introduce in Section 4 and, perhaps, illustrates their utility. For now, it is enough to know that given a finite dimensional normed space X , there exists a constant $\alpha(X)$ called the *alpha constant* of the space, with the property that

$$\|\phi\|_{cb} \leq \alpha(X)\|\phi\|$$

for any map with domain an operator space that is isometrically isomorphic to X as normed spaces. Given two finite dimensional normed spaces X, Y of the same dimension one has

$$\alpha(X) \leq d(X, Y)\alpha(Y),$$

where $d(X, Y)$ denotes the *Banach-Mazur distance* between the spaces. These concepts and results can be found in [22].

Theorem 6. *Let \mathcal{M} be an operator space and let $\phi : M_n \rightarrow \mathcal{M}$ be a linear map, then $\|\phi\|_{cb} \leq n\sqrt{n}\|\phi\|$.*

Proof. Let $\|X\|_2$ denote the Hilbert-Schmidt norm of a matrix X . Since $\|X\| \leq \|X\|_2 \leq \sqrt{n}\|X\|$, the Banach-Mazur distance satisfies $d(M_n, \mathbb{C}^{n^2}) \leq \sqrt{n}$.

Hence, $\alpha(M_n) \leq d(M_n, \mathbb{C}^{n^2})\alpha(\mathbb{C}^{n^2}) \leq \sqrt{n}\alpha(\mathbb{C}^{n^2})$. Finally, it is shown in [22], that for Euclidean space, $\alpha(\mathbb{C}^m) \leq \sqrt{m}$, from which the result follows. \square

Combining this result with Theorem 3 we have:

Corollary 7. *Let $\phi : M_n \rightarrow M_k$ be a linear map, then*

$$\|\phi\|_{cb} \leq \min\{k, \sqrt{n^3}\}\|\phi\|.$$

We next recall the abstract definition of completely positive maps.

Definition 8. *If \mathcal{A} is a unital C^* -algebra, then a \dagger -closed subspace $\mathcal{S} \subseteq \mathcal{A}$ such that $1 \in \mathcal{S}$, is called an **operator system**.*

Thus, operator systems are operator spaces and have matrix norms. But the additional hypotheses guarantee that if we let \mathcal{A}^+ denote the positive elements of the C^* -algebra, then \mathcal{S} is the span of $\mathcal{S}^+ \equiv \mathcal{S} \cap \mathcal{A}^+$, which is a cone in \mathcal{S} . We also have that $M_n(\mathcal{S})$ is the span of $M_n(\mathcal{S})^+ = M_n(\mathcal{S}) \cap M_n(\mathcal{A})^+$. The vector spaces, $M_n(\mathcal{S})$ together with the cones $M_n(\mathcal{S})^+$ is often referred to as the *matrix ordering* on \mathcal{S} .

Definition 9. *Given a unital C^* -algebra \mathcal{A} , $\mathcal{S} \subseteq \mathcal{A}$ and a map $\phi : \mathcal{S} \rightarrow B(\mathcal{H})$, we call ϕ **completely positive**, provided that ϕ_n is positive for all n , that is provided that $(a_{i,j}) \in M_n(\mathcal{S})^+$ implies that $(\phi(a_{i,j})) \in M_n(B(\mathcal{H}))^+$.*

Definition 10. *Given a C^* -algebra \mathcal{A} and $\mathcal{M} \subseteq \mathcal{A}$ an operator space, we set $\mathcal{M}^* = \{a^\dagger : a \in \mathcal{M}\}$, which is another operator space. If $\phi : \mathcal{M} \rightarrow B(\mathcal{H})$, is a linear map, then we define $\phi^* : \mathcal{M}^* \rightarrow B(\mathcal{H})$ by $\phi^*(b) = \phi(b^\dagger)^\dagger$, which is another linear map.*

The following objects allow one to relate much of the theory of completely bounded maps to the more familiar theory of completely positive maps.

Definition 11. *Let \mathcal{A} be a unital C^* -algebra and let $\mathcal{M} \subseteq \mathcal{A}$ be an operator space, then we define an operator system $\mathcal{S}_{\mathcal{M}} \subseteq M_2(\mathcal{A})$, by*

$$\mathcal{S}_{\mathcal{M}} \equiv \left\{ \begin{pmatrix} \lambda 1 & a \\ b^\dagger & \mu 1 \end{pmatrix} : \lambda \in \mathbb{C}, \mu \in \mathbb{C}, a \in \mathcal{M}, b \in \mathcal{M} \right\}.$$

Details for the next six results can be found in [21]. We briefly sketch some of the ideas of the proofs to help indicate the interplay between completely bounded maps and completely positive maps.

Theorem 12. *Let \mathcal{A} be a unital C^* -algebra, $\mathcal{M} \subseteq \mathcal{A}$ be an operator space and let $\phi : \mathcal{M} \rightarrow B(\mathcal{H})$ be linear. Then $\|\phi\|_{cb} \leq 1$ if and only if $\Phi : \mathcal{S}_{\mathcal{M}} \rightarrow M_2(B(\mathcal{H}))$ is completely positive, where*

$$\Phi\left(\begin{pmatrix} \lambda 1 & a \\ b^\dagger & \mu 1 \end{pmatrix}\right) = \begin{pmatrix} \lambda I_{\mathcal{H}} & \phi(a) \\ \phi(b)^\dagger & \mu I_{\mathcal{H}} \end{pmatrix}.$$

In particular, using this identification of completely contractive ($\|\phi\|_{cb} \leq 1$) maps with ‘‘corners’’ of unital completely positive maps, one extension theorem:

Theorem 13 (Arveson’s Extension Theorem). *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system and let $\phi : \mathcal{S} \rightarrow B(\mathcal{H})$ be completely positive. Then there exists a completely positive map $\psi : \mathcal{A} \rightarrow B(\mathcal{H})$ that extends ϕ , that is, such that $\psi(a) = \phi(a)$ for every $a \in \mathcal{S}$.*

quickly yields another:

Theorem 14 (Wittstock’s Extension Theorem). *Let $\mathcal{M} \subseteq \mathcal{A}$ be an operator space and let $\phi : \mathcal{M} \rightarrow B(\mathcal{H})$ be completely bounded. Then there exists a completely bounded map $\psi : \mathcal{A} \rightarrow B(\mathcal{H})$ that extends ϕ and satisfies $\|\psi\|_{cb} = \|\phi\|_{cb}$.*

To obtain the second from the first, one first scales ϕ so that $\|\phi\|_{cb} = 1$, then applies Arveson’s Theorem to extend $\Phi : \mathcal{S}_{\mathcal{M}} \rightarrow M_2(B(\mathcal{H}))$, to $\Psi : M_2(\mathcal{A}) \rightarrow M_2(B(\mathcal{H}))$, and then lets ψ be the corresponding (1,2) matrix corner of Ψ .

A fundamental result for quantum information is Stinespring’s classical representation theorem for completely positive maps.

Theorem 15 (Stinespring’s Representation Theorem). *Let \mathcal{A} be a unital C^* -algebra and let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be a completely positive map, then there exists a Hilbert space \mathcal{K} , a bounded operator $V : \mathcal{H} \rightarrow \mathcal{K}$ and a unital \dagger -homomorphism, $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$ such that $\phi(a) = V^\dagger \pi(a) V$, for every $a \in \mathcal{A}$.*

Note that in Stinespring’s theorem, we also have that $\|\phi\|_{cb} = \|\phi(1)\| = \|V^\dagger V\| = \|V\|^2$.

This form of the Stinespring Theorem is less common in quantum information, but the more standard forms can be readily obtained. Suppose $\phi : M_n \rightarrow M_k$ is a completely positive unital map. As every representation of M_n is unitarily equivalent to a multiple of the identity representation [8], π can be assumed to be of the form $\pi(a) = I_E \otimes a$, where I_E is the identity operator on a suitable dilation Hilbert space E . Further, as ϕ is unital we have $I = \phi(I) = V^\dagger V$, and hence V is an isometry. Thus we may write ϕ as $\phi(a) = V^\dagger (I_E \otimes a) V$. The dual of this equation yields the familiar partial trace form for a quantum channel, $\phi^\dagger(a) = \text{Tr}_E(V a V^\dagger)$.

In a similar fashion to the extension theorem, Stinespring's Theorem can be extended to completely bounded maps.

Theorem 16 (The Generalized Stinespring Theorem). *Let \mathcal{A} be a unital C^* -algebra and let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be a completely bounded map, then there exists a Hilbert space \mathcal{K} , bounded operators $V : \mathcal{H} \rightarrow \mathcal{K}, W : \mathcal{H} \rightarrow \mathcal{K}$ and a unital \dagger -homomorphism, $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$, such that $\|\phi\|_{cb} = \|V\|\|W\|$ and $\phi(a) = V^\dagger \pi(a)W$, for every $a \in \mathcal{A}$.*

In the finite dimensional case of a completely bounded map $\phi : M_n \rightarrow M_k$, the corresponding canonical forms look like $\phi(a) = V^\dagger(I_E \otimes a)W$ and $\phi^\dagger(a) = \text{Tr}_E(WaV^\dagger)$.

The generalization of Stinespring's theorem to completely bounded maps also yields the following "polar form" for completely bounded maps. To motivate this result note that for operators, if we set $|T| = \sqrt{T^\dagger T}$, then $\begin{pmatrix} |T^\dagger| & T \\ T^\dagger & |T| \end{pmatrix}$ is positive.

Corollary 17. *Let \mathcal{A} be a unital C^* -algebra and let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be completely bounded, then there exists completely positive maps, $\phi_1, \phi_2 : \mathcal{A} \rightarrow B(\mathcal{H})$, with $\|\phi_1(1)\| = \|\phi_2(1)\| = \|\phi\|_{cb}$, such that $\Phi : M_2(\mathcal{A}) \rightarrow M_2(B(\mathcal{H}))$ is completely positive, where*

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \phi_1(a) & \phi(b) \\ \phi^*(c) & \phi_2(d) \end{pmatrix}.$$

When $\|\phi\|_{cb} \leq 1$, then ϕ_1, ϕ_2 can both be taken to be unital completely positive maps. Thus, the above corollary is another example of the meta-theorem that completely contractive maps are the "corners" of unital completely positive maps.

The generalized Stinespring theorem, unfortunately, has no good uniqueness criteria, unlike the usual Stinespring theorem. The difficulty stems from the fact that the two completely positive maps, ϕ_1, ϕ_2 are not uniquely determined by ϕ . Generally, there are many possible extensions of the completely positive map Φ from the operator system $\mathcal{S}_{\mathcal{A}}$ to $M_2(\mathcal{A})$ and this allows for a great deal of non-uniqueness. Ostensibly this follows from the fact that the Hahn-Banach Theorem plays a key role in the proof. Thus, in particular, the above "polar form" of a completely bounded map is not unique.

Just as one obtains the Choi-Kraus representation of completely positive maps from M_n to M_k by specializing Stinespring's theorem to these algebras, one obtains a similar representation of completely bounded maps.

Theorem 18 (Choi-Kraus Representation Theorem [5, 17]). *Let $\phi : M_n \rightarrow M_k$ be completely positive, then there exists matrices, $A_i \in M_{n,k}, 1 \leq i \leq nk$, such that $\phi(X) = \sum_i A_i^\dagger X A_i$.*

Theorem 19 (CB Representation Theorem). *Let $\phi : M_n \rightarrow M_k$ be a linear map. Then there exists matrices, $A_i \in M_{k,n}, 1 \leq i \leq m$, and matrices $B_i \in M_{n,k}, 1 \leq i \leq m$, such that*

$$(1) \quad \phi(X) = \sum_i A_i X B_i,$$

with $\|\phi\|_{cb}^2 = \|\phi^\dagger\|_\diamond^2 = \|\sum_i A_i A_i^\dagger\| \|\sum_i B_i^\dagger B_i\|$ and $m \leq nk$.

It is important to understand the difference between ϕ^* and the usual dual map ϕ^\dagger considered in quantum information, so let us dwell on this point for a moment with $\mathcal{M} = M_n, B(\mathcal{H}) = M_k$, and $\phi : M_n \rightarrow M_k$. In terms of Choi-Kraus operation elements, if $A \in M_{k,n}, B \in M_{n,k}$, and $\phi : M_n \rightarrow M_k$, is defined by $\phi(X) = AXB$, then $\phi^* : M_n \rightarrow M_k$ is given by $\phi^*(X) = B^* X A^*$, while $\phi^\dagger : M_k \rightarrow M_n$ is given by $\phi^\dagger(Y) = BYA$, and the obvious generalization holds true if ϕ is given by a sum of such terms.

Another difference between ϕ^* and ϕ^\dagger arises when considering the CB norm. It is easily checked that $\|\phi_n\| = \|(\phi^*)_n\|$ and hence that $\|\phi\|_{cb} = \|\phi^*\|_{cb}$. On the other hand, $\|\phi\|_{cb}$ and $\|\phi^\dagger\|_{cb}$ can be different. For instance, in the case of a completely positive, trace preserving map ϕ , the dual ϕ^\dagger is unital ($\phi^\dagger(I) = I$), so that $\|\phi^\dagger\|_{cb} = \|\phi^\dagger(I)\| = 1$, whereas $\|\phi\|_{cb} = \|\phi(I)\|$ could be larger or smaller.

We shall call *any* representation $\phi(X) = \sum_i A_i X B_i$ a *generalized Choi-Kraus representation*. Note that if we have any generalized Choi-Kraus representation of ϕ then,

$$\phi(X) = (A_1, \dots, A_m) \begin{pmatrix} X & 0 & \dots & 0 \\ 0 & X & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X \end{pmatrix} \begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix},$$

where the term in the middle represents the $m \times m$ block diagonal matrix each of whose blocks is X and hence,

$$\|\phi\|_{cb} \leq \|(A_1, \dots, A_m)\| \left\| \begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix} \right\| = \left\| \sum_i A_i A_i^\dagger \right\|^{1/2} \left\| \sum_i B_i^\dagger B_i \right\|^{1/2},$$

which explains the asymmetry in the roles of the A's and B's.

This also leads to the following very useful result.

Corollary 20. *Let $\phi : M_n \rightarrow M_k$ be a linear map, then*

$$\|\phi\|_{cb} = \|\phi^\dagger\|_\diamond = \inf \left\{ \left\| \sum_i A_i A_i^\dagger \right\|^{1/2} \left\| \sum_i B_i^\dagger B_i \right\|^{1/2} \right\},$$

where the infimum is taken over all generalized Choi-Kraus representations of ϕ .

4. COMPUTATION AND ESTIMATION OF THE CB/ \diamond NORM

In this section we present the theoretical formulation of the algorithm and use it to derive a geometric formula for mixtures of pairs of unitary maps. In the case of a completely positive map ϕ , Theorem 19 coupled with the Choi-Kraus representation theorem shows that the CB norm of ϕ is exactly the operator norm $\|\phi(I)\|$. For completeness we provide the direct, elementary proof of this fact from [21] with no restriction on the domain of the map.

Lemma 21. *Let P and A be operators on some Hilbert space \mathcal{H} with P positive. Then $\begin{pmatrix} P & A \\ A^* & P \end{pmatrix} \geq 0$ implies that $\|A\| \leq \|P\|$. Furthermore, if P is the identity operator then the converse also holds.*

Proof. To show the forward implication, note that if $\begin{pmatrix} P & A \\ A^* & P \end{pmatrix} \geq 0$ then it follows that $\left\langle \begin{pmatrix} P & A \\ A^* & P \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} \middle| \begin{pmatrix} x \\ -y \end{pmatrix} \right\rangle \geq 0 \forall x, y \in \mathcal{H}$ s.t. $\|x\| = \|y\| = 1$. Thus, $\langle Px|x \rangle + \langle Py|y \rangle \geq \langle Ay|x \rangle + \langle x|Ay \rangle = 2\text{Re}(\langle Ay|x \rangle)$. Also, the Cauchy-Schwarz Inequality tells us that $\langle Px|x \rangle + \langle Py|y \rangle \leq \|Px\| + \|Py\| \leq 2\|P\|$ since $\|x\| = \|y\| = 1$. Thus, $\|P\| \geq \text{Re}(\langle Ay|x \rangle) \forall x, y \in \mathcal{H}$ s.t. $\|x\| = \|y\| = 1$. A little thought reveals that this is equivalent to $\|P\| \geq \langle Ay|x \rangle \forall x, y \in \mathcal{H}$ s.t. $\|x\| = \|y\| = 1$, which immediately implies that $\|A\| \leq \|P\|$.

To show the converse when $P = I$ is the identity operator, we prove by contradiction by assuming that $\|A\| \leq \|I\| = 1$ and that $\exists x, y \in \mathcal{H}$ such that $\left\langle \begin{pmatrix} I & A \\ A^* & I \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} \middle| \begin{pmatrix} x \\ -y \end{pmatrix} \right\rangle < 0$. It then follows that $\|x\|^2 + \|y\|^2 < \langle Ay|x \rangle + \langle x|Ay \rangle$.

The Cauchy-Schwarz Inequality tells us that $\langle Ay|x \rangle + \langle x|Ay \rangle \leq \|Ay\| \|x\| + \|x\| \|Ay\| \leq 2\|A\| \|x\| \|y\| \leq 2\|x\| \|y\|$. Thus, $\|x\|^2 + \|y\|^2 < 2\|x\| \|y\|$. This, however, is impossible since $\forall x, y \in \mathcal{H}$ it is true that $(\|x\|^2 + \|y\|^2)^2 = (\|x\|^2 - \|y\|^2)^2 + 4\|x\|^2 \|y\|^2 \geq 4\|x\|^2 \|y\|^2$, so $\|x\|^2 + \|y\|^2 \geq 2\|x\| \|y\|$, completing the contradiction. \square

Theorem 22. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, let \mathcal{B} be a C^* -algebra, and let $\phi : \mathcal{S} \rightarrow \mathcal{B}$ be a completely positive map. Then ϕ is completely bounded and $\|\phi\|_{cb} = \|\phi\| = \|\phi(1)\|$.*

Proof. First note that $\|\phi(1)\| \leq \|\phi\| \leq \|\phi\|_{cb}$, so we need only show that $\|\phi\|_{cb} \leq \|\phi(1)\|$.

Fix n and let $A \in M_n(\mathcal{S})$ be such that $\|A\| \leq 1$, and let I_n be the unit of $M_n(\mathcal{A})$. Then Lemma 21 tells us that $\begin{pmatrix} I_n & A \\ A^* & I_n \end{pmatrix} \geq 0$. Since ϕ is completely positive, it then follows that $\phi_{2n} \begin{pmatrix} I_n & A \\ A & I_n \end{pmatrix} = \begin{pmatrix} \phi_n(I_n) & \phi_n(A) \\ \phi_n(A)^* & \phi_n(I_n) \end{pmatrix} \geq 0$. Making use of Lemma 21 again shows us that $\|\phi_n(A)\| \leq \|\phi_n(I_n)\| = \|\phi(1)\|$. Since this inequality holds for any such A , the proof is complete. \square

4.1. The Algorithm. We now turn to the problem of actually computing the norm of a linear map $\phi : M_n \rightarrow M_k$. By the above results we know that to compute the cb-norm we need to do a minimization over all generalized Choi-Kraus representations. This turns out to be somewhat more attainable than might be imagined and we present an algorithm for computing the cb/ \diamond -norm of such maps. We first describe the algorithm and then justify it later.

We assume that we are given a map $\phi : M_n \rightarrow M_k$, some generalized Choi-Kraus representation $\phi(X) = \sum_{i=1}^m A_i X B_i$ and we wish to compute $\|\phi\|_{cb} = \|\phi^\dagger\|_\diamond$.

Step 1. Find a basis, $\{C_1, \dots, C_l\}$ for the span of $\{B_1, \dots, B_m\}$ and express $B_i = \sum d_{i,j} C_j$

Step 2. Using the expressions for each B_i as a linear combination of C_j we may re-write $\phi(X) = \sum_{j=1}^l D_j X C_j$. In fact, we have $\phi(X) = \sum_i A_i X (\sum_j d_{i,j} C_j) = \sum_j (\sum_i d_{i,j} A_i) X C_j$. Thus,

$$D_j = \sum_i d_{i,j} A_i.$$

Step 3. Find a basis $\{E_1, \dots, E_p\}$ for the span of $\{D_1, \dots, D_l\}$, express each D_j as a linear combination, and repeat Step 2 to obtain

$$\phi(X) = \sum_{i=1}^p E_i X F_i,$$

where the F_i 's are the corresponding linear combinations of the C_j 's.

Remarkably, at this stage it is a theorem that the sets $\{E_1, \dots, E_p\}$ and $\{F_1, \dots, F_p\}$ are linearly independent, and hence this process terminates.

Step 4. Given an invertible $S = (s_{i,j}) \in M_p$ with inverse $S^{-1} = (t_{i,j}) \in M_p$, let $H_i = \sum_j s_{i,j} F_j$, and $G_j = \sum_i t_{i,j} E_i$. Then

$$\|\phi\|_{cb} = \inf \left\{ \left\| \sum_i G_i G_i^* \right\|^{1/2} \left\| \sum_i H_i^* H_i \right\|^{1/2} \right\},$$

where the infimum is taken over all invertible matrices S . It is also enough to consider positive, invertible matrices for S .

This algorithm reduces the computation of $\|\phi\|_{cb}$ to a series of matrix computations and only the last step might involve a difficult minimization.

To begin to justify the algorithm, we begin with the last step. First we show that $\phi(X) = \sum_i G_i X H_i$. This can be seen formally, because

$$\begin{aligned} \sum_i G_i X H_i &= (G_1, \dots, G_p) \begin{pmatrix} X & 0 & \dots & 0 \\ 0 & X & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X \end{pmatrix} \begin{pmatrix} H_1 \\ \vdots \\ H_p \end{pmatrix} \\ &= (E_1, \dots, E_p) (t_{i,j} I_n) \begin{pmatrix} X & 0 & \dots & 0 \\ 0 & X & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X \end{pmatrix} (s_{i,j} I_n) \begin{pmatrix} F_1 \\ \vdots \\ F_p \end{pmatrix} \\ &= \sum_i E_i X F_i = \phi(X), \end{aligned}$$

since the two scalar matrices commute past the direct sum in the middle. Note that these scalar matrices behave like “environmental operators”, that is, they are operators that act exclusively on the environment of an open quantum system.

Next we need to see that the linear maps from M_n to M_k , which we denote by $\mathcal{L}(M_n, M_k)$, can be identified with the tensor product, $M_{k,n} \otimes M_{n,k}$ via the map that sends an elementary tensor, $A \otimes B$ to the map $\phi(X) = AXB$. It is easily seen that this extends to a linear map, $\Gamma : M_{k,n} \otimes M_{n,k} \rightarrow \mathcal{L}(M_n, M_k)$, that a simple dimension count shows is one-to-one and onto (both spaces have dimension $n^2 k^2$).

We now endow $M_{k,n} \otimes M_{n,k}$ with a norm so that Γ will be an isometry when $\mathcal{L}(M_n, M_k)$ is endowed with the cb-norm. By the CB representation theorem, we see that if we define for $U \in M_{k,n} \otimes M_{n,k}$,

$$\|U\|_h = \inf \left\{ \left\| \sum_i A_i A_i^\dagger \right\|^{1/2} \left\| \sum_i B_i^\dagger B_i \right\|^{1/2} \right\},$$

where the infimum is taken over all ways to represent $U = \sum_i A_i \otimes B_i$ as a sum of elementary tensors, then we will have that $\|U\|_h = \|\Gamma(U)\|_{cb}$.

The above tensor norm is called the *Haagerup tensor norm* in honor of U. Haagerup who was the first to notice the above identification. We write $M_{k,n} \otimes_h M_{n,k}$ to denote the tensor product endowed with this norm and note that we have just proved that:

Theorem 23 (Haagerup). *The map $\Gamma : M_{k,n} \otimes_h M_{n,k} \rightarrow CB(M_n, M_k)$ defined by $\Gamma(A \otimes B)(X) = AXB$ is an isometric isomorphism.*

Here we use $CB(M_n, M_k)$ to denote the space of linear maps from M_n to M_k endowed with the completely bounded norm. The above isomorphism was greatly extended in work of Haagerup and Effros-Kishimoto to other identifications between spaces of completely bounded maps and Haagerup tensor products.

The above theorem reduces the justification of the above algorithm to showing that if $\phi = \Gamma(U)$, then the algorithm correctly computes, $\|U\|_h$. The fact that this algorithm correctly computes $\|U\|_h$ for any operator spaces is proven in [3]. We outline the key ideas below. For this, we will need a few facts about tensor products of vector spaces.

Recall that if \mathcal{V} and \mathcal{W} are vector spaces, then every element of $\mathcal{V} \otimes \mathcal{W}$ is a finite sum of elementary tensors. The least number of elementary tensors that can be used to represent an element $u \in \mathcal{V} \otimes \mathcal{W}$ is called the *rank of u* and is denoted by $\text{rank}(u)$.

Proposition 24. [3] *Let $u \in \mathcal{V} \otimes \mathcal{W}$. If $u = \sum_{i=1}^p v_i \otimes w_i$ then $p = \text{rank}(u)$ if and only if $\{v_1, \dots, v_p\}$ is a linearly independent set and $\{w_1, \dots, w_p\}$ is a linearly independent set. Moreover, if $u = \sum_{i=1}^p x_i \otimes y_i$ is another way to represent u as a sum of elementary tensors and $p = \text{rank}(u)$, then*

$$\text{span}\{v_1, \dots, v_p\} = \text{span}\{x_1, \dots, x_p\}$$

and

$$\text{span}\{w_1, \dots, w_p\} = \text{span}\{y_1, \dots, y_p\}.$$

Proposition 25. [3] *Let $u \in \mathcal{V} \otimes \mathcal{W}$. If we apply Step 1 and Step 2 of the above algorithm to $u = \sum_{i=1}^m a_i \otimes b_i$, to obtain $u = \sum_{i=1}^p e_i \otimes f_i$, then $\{e_1, \dots, e_p\}$ and $\{f_1, \dots, f_p\}$ will be linearly independent sets and hence $\text{rank}(u) = p$.*

These facts are easily proved by applying maps of the form $f \otimes id_{\mathcal{W}}$ and $id_{\mathcal{V}} \otimes g$, where f and g are linear functionals to u .

The remainder of the proof of the justification of the algorithm is to show that at each stage, removing the linear dependencies among the elements in the sum for u reduces the Haagerup norm. This is best seen at each stage of the algorithm. Say at Step 1, when we choose the basis, $\{C_1, \dots, C_l\}$ and express, $B_i = \sum_j d_{i,j} C_j$, if we first polar decompose the matrix $(d_{i,j}) = (w_{i,j})(p_{i,j})$ where $W = (w_{i,j})$ is an $m \times l$ partial isometry and $P = (p_{i,j})$ is an invertible $l \times l$ positive matrix, then we have that $B_i = \sum_j w_{i,j} \tilde{C}_j$, with $\tilde{C}_i = \sum_j p_{i,j} C_j$. In this case, the set $\{\tilde{C}_1, \dots, \tilde{C}_l\}$ is another basis for the span of $\{C_1, \dots, C_l\}$ and $\sum_i \tilde{C}_i^\dagger \tilde{C}_i = \sum_i C_i^\dagger C_i$. Moreover, using this basis, we would obtain another representation for $\phi(X) = \sum_{j=1}^l \tilde{D}_j X \tilde{C}_j$, where $\tilde{D}_j = \sum_i w_{i,j} A_i$. Again, since P is invertible, the span of $\{\tilde{D}_1, \dots, \tilde{D}_l\}$ is the same as the span of $\{D_1, \dots, D_l\}$. Moreover, since W is a partial isometry, one finds that $\sum_i \tilde{D}_i \tilde{D}_i^\dagger \leq \sum_i A_i A_i^\dagger$. Thus, the infimum of the Haagerup norm expression over all linear combinations of the D_i 's and C_i 's which is the same as the infimum over all linear combinations of the \tilde{D}_i 's and \tilde{C}_i 's is smaller.

This proves that the quantity defining the Haagerup tensor norm (which is the same as the CB norm) must be attained when the coefficients of the generalized Choi-Kraus representation are linearly independent, and hence represented by some choice of basis for $\text{span}\{E_1, \dots, E_p\}$ and $\text{span}\{F_1, \dots, F_p\}$.

4.2. Example. In quantum information, maps given by the difference of two (distinct) unitary maps form the most elementary class of linear, non-completely positive maps of interest. We next show how the algorithm can be used to derive a simple geometric technique that computes the exact stabilized norm for maps in this class. We note this result can be derived from a technical result of Herrero ([13], Theorem 3.31), which is proved using operator theoretic machinery. Moreover, the result is also stated more recently in [1] without proof. Our proof is new and elementary and gives a good illustration of the algorithm at work. By the unitary invariance of the cb/\diamond norm, observe that we can compute the norm of any map $\mathcal{U} - \mathcal{V}$ once we know how to compute it for any map of the form $\mathcal{U} - \text{id}$.

Theorem 26. *Let $U \in M_n$ be a unitary operator on a finite dimensional Hilbert space and let $\Phi : M_n \rightarrow M_n$ be given by $\Phi(X) = UXU^\dagger - X$. Then $\|\Phi\|_{cb} = \|\Phi^\dagger\|_{cb} = \|\Phi\|_\diamond = \|\Phi^\dagger\|_\diamond$ is equal to the diameter of the smallest closed disc that contains all of the eigenvalues of U .*

Proof. If U is a scalar multiple of I then $\Phi \equiv 0$ so the result immediately follows. Thus, we will assume from here on that U is not a scalar multiple of I . It then follows from Step 4 of the algorithm that

$$\|\Phi\|_{cb} = \inf \left\{ \left\| (U^\dagger, I) \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right\| \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} U \\ -I \end{pmatrix} \right\| \right\}$$

where the infimum is over all invertible 2×2 scalar matrices.

Now let $\mathbf{v} = (a \ c)^T$ and $\mathbf{w} = (b \ d)^T$ so that

$$\begin{aligned} \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} U \\ -I \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} aU - bI \\ cU - dI \end{pmatrix} \right\|^2 \\ &= \left\| (|a|^2 + |b|^2 + |c|^2 + |d|^2)I - (\bar{a}b + \bar{c}d)U^\dagger - (a\bar{b} + c\bar{d})U \right\|^2 \\ &= \left\| (\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)I - 2\text{Re}(\langle \mathbf{v}, \mathbf{w} \rangle U) \right\|^2 \end{aligned}$$

If we let $D = ad - bc$ be the determinant of the matrix, then a similar calculation shows that

$$\left\| (U^\dagger, I) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} D^{-1} \right\|^2 = |D|^{-2} \left\| (\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)I - 2\text{Re}(\langle \mathbf{v}, \mathbf{w} \rangle U) \right\|^2$$

Thus it follows that

$$\|\Phi\|_{cb} = \inf \{ |D|^{-1} \| (\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)I - 2\text{Re}(\langle \mathbf{v}, \mathbf{w} \rangle U) \| \}$$

where the infimum is taken over all 2×1 complex vectors \mathbf{v} and \mathbf{w} .

Now it is clear that this minimum will be attained when \mathbf{v} and \mathbf{w} are rotated such that $\min_i \{ \text{Re}(\langle \mathbf{v}, \mathbf{w} \rangle \lambda_i) \}$ is as large as possible (while keeping \mathbf{v} and \mathbf{w} of fixed length), where λ_i ranges over all eigenvalues of U . Thus, since multiplying \mathbf{w} by $e^{i\alpha}$ will not change $|D|$, it follows that

$$\|\Phi\|_{cb} = \inf \{ |D|^{-1} \| (\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)I - 2\text{Re}(\langle \mathbf{v}, \mathbf{w} \rangle e^{i\alpha} U) \| \}$$

where α is such that the minimum real part of the eigenvalues of $e^{i\alpha}U$ is as large as possible. Define r to be this largest minimum real eigenvalue part.

Now, similar to before, we can multiply \mathbf{w} by $e^{i\beta}$ so that $|\langle \mathbf{v}, \mathbf{w} \rangle| = \langle \mathbf{v}, e^{i\beta} \mathbf{w} \rangle$, and so it follows that

$$\|\Phi\|_{cb} = \inf\{|D|^{-1}(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)I - 2|\langle \mathbf{v}, \mathbf{w} \rangle| rI\}$$

where the infimum is now taken over all 2×1 real vectors \mathbf{v} and \mathbf{w} . It is now clear that we can assume without loss of generality that $\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 = 1$. It also follows from some simple algebra that, given any two vectors \mathbf{v} and \mathbf{w} such that $\|\mathbf{v}\| \neq \|\mathbf{w}\|$, the value within this infimum will be made smaller by scaling \mathbf{v} and \mathbf{w} so that $\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2 = \frac{1}{2}$.

It then immediately follows from expanding out the terms within the infimum that this is equivalent to the following minimization problem

$$\|\Phi\|_{cb} = \min \left\{ \frac{1 - 2|ab + cd|r}{|ad - bc|} \right\}$$

such that $a^2 + c^2 = b^2 + d^2 = \frac{1}{2}$

Now, if $r \leq 0$ then it is easy to see that this minimum is equal to 2 by setting $a = d = \frac{1}{\sqrt{2}}$ and $b = c = 0$. Thus, it only remains to prove the conjecture in the case when $r > 0$. If $r > 0$ then it is clear that this minimization problem is equivalent to the one we get if we remove the absolute value bars in the numerator.

We now form the Lagrangian of this problem:

$$\Lambda = \frac{1 - 2(ab + cd)r}{|ad - bc|} + \lambda_1 \left(a^2 + c^2 - \frac{1}{2} \right) + \lambda_2 \left(b^2 + d^2 - \frac{1}{2} \right)$$

If we now set $\frac{\partial \Lambda}{\partial b} = \frac{\partial \Lambda}{\partial d}$, we arrive at the equation

$$\langle \mathbf{v}, \mathbf{w} \rangle = ab + cd = 2(a^2 + c^2)(b^2 + d^2)r = \frac{r}{2}$$

This, however, implies that $\theta = \arccos(r)$, where θ is the angle between \mathbf{v} and \mathbf{w} . Thus, this problem is minimized by vectors \mathbf{v} and \mathbf{w} that are each of length $\frac{1}{\sqrt{2}}$ and separated by an angle $\arccos(r)$. This, however, implies that $|D| = \|\mathbf{v}\| \|\mathbf{w}\| \sin \theta = \frac{1}{2} \sqrt{1 - r^2}$. Plugging this and $ab + cd = \frac{r}{2}$ into the formula to be minimized, we learn that

$$\|\Phi\|_{cb} = 2\sqrt{1 - r^2}$$

It now is a simple geometric argument that finally shows that this value is equal to the diameter of the smallest closed disc enclosing the eigenvalues of U , completing the proof. \square

5. IMPLEMENTATION OF THE ALGORITHM

The implementation of the algorithm *via* Maple that we now present is split up into several procedures, which will be described as we present their code. We have kept variable names within the code as close as possible to their counterparts presented in the above theoretical discussion of the algorithm. After the code has been presented and briefly explained, we discuss its efficiency and provide an example that shows how the code is used, comparing the results of the algorithm to known theoretical results.

The first procedure, `RandomPositive`, generates a random positive matrix with eigenvalues in the interval $(\text{evalLower}, \text{evalUpper}]$. This is achieved by generating a diagonal matrix with entries contained in that interval and conjugating by a random unitary. The random unitary matrix is constructed by generating a matrix with random entries from the square with corners at 0 and $1 + i$ and then using the Gram-Schmidt process on its columns.

```
> RandomPositive := proc(ndim,evalLower,evalUpper)
  local r,i,RD,RM,V,W,U,P:r:=0:
  RD:=RandomMatrix(ndim,ndim,generator=evalLower+
    DBL_EPSILON..evalUpper,outputoptions=[shape=diagonal]):

  while r < ndim do
    RM:=RandomMatrix(ndim,ndim,generator=0.0..1.0) +
      I*RandomMatrix(ndim,ndim,generator=0.0..1.0):
    r:=round(Rank(RM)):
  od:
  for i from 1 to ndim do
    V[i]:=Vector(ndim,(j) -> RM[j,i]):
  od:
  W:=GramSchmidt(convert(V,list),normalized):
  U:=Matrix(ndim,ndim,(i,j) -> W[j][i]):
  P[1]:=MatrixMatrixMultiply(MatrixMatrixMultiply(U,RD),
    HermitianTranspose(U)):
  P[2]:=MatrixMatrixMultiply(MatrixMatrixMultiply(U,
    MatrixInverse(RD)),HermitianTranspose(U)):
  RETURN(P[1],P[2]):
end:
```

The procedure `IsCPMap` determines whether or not the completely bounded map that it is given is actually a completely positive map by determining whether or not its Choi matrix is positive. This procedure is optional for the algorithm; it simply serves to allow the algorithm to compute the completely bounded norm of completely positive maps more quickly and more accurately than it otherwise could.

```
> IsCPMap := proc(CelA,CelB,NumOps,n,k)
  local i,j,x,Choi,LgChoi,MtxUnit:
```

```

if not (n = k) then
  RETURN(false):
else
  for i from 1 to n do
    for j from 1 to n do
      MtxUnit[i][j]:=OuterProductMatrix(UnitVector(i,n),
        UnitVector(j,n)):
      Choi[i][j]:=add(MatrixMatrixMultiply(
        MatrixMatrixMultiply(A[x],MtxUnit[i][j]),B[x]),
        x=1..NumOps):
    od:
  od:

  LgChoi:=Matrix(n^2,n^2,(i,j) -> Choi[floor((i-1)/n)+1]
    [floor((j-1)/n)+1][((i-1) mod n)+1,((j-1) mod n)+1]):
  RETURN(IsDefinite(LgChoi)):
fi:
end:

```

The procedures `MakeLinIndep` and `CellMatricize` jointly perform steps 1 through 3 of the algorithm. They are reasonably straightforward.

```

> MakeLinIndep := proc(CelA,CelB,NumOps,n,k)
  local BM,BS,u,x,i,j,bg,BG,d,C,DOp,CM,CS,v,cg,CG,c,E,F:
  BM:=CellMatricize(CelB,NumOps,n,k):u:=round(Rank(BM)):
  BS:=Basis({Row(BM,[1..NumOps])}):
  for x from 1 to NumOps do
    for i from 1 to n*k do
      for j from 1 to u do
        bg[i,j] := BS[j][i]:
      od:
      bg[i,u+1] := BM[x,i]:
    od:
    BG[x]:=ReducedRowEchelonForm(
      Matrix(n*k,u+1,(i,j) -> bg[i,j])):

    for j from 1 to u do
      d[x,j]:=BG[x][j,u+1]:
    od:
  od:

  for x from 1 to u do
    C[x]:=Matrix(k,n,(i,j) ->
      add(d[1,x]*CelA[1][i,j],l=1..NumOps)):
    DOp[x]:=Matrix(n,k,(i,j) -> BS[x][j+(i-1)*k]):
  end:

```

```

od:

CM:=CellMatricize(C,u,k,n):v:=round(Rank(CM)):
CS:=Basis({Row(CM,[1..u])}):
for x from 1 to u do
  for i from 1 to n*k do
    for j from 1 to v do
      cg[i,j] := CS[j][i]:
    od:
    cg[i,v+1] := CM[x,i]:
  od:
  CG[x]:=ReducedRowEchelonForm(
    Matrix(n*k,v+1,(i,j) -> cg[i,j])):

  for j from 1 to v do
    c[x,j]:=CG[x][j,v+1]:
  od:
od:

for x from 1 to v do
  E[x]:=Matrix(k,n,(i,j)->CS[x][j+(i-1)*n]):
  F[x]:=Matrix(n,k,(i,j)->add(c[l,x]*D0p[l][i,j],l=1..u)):
od:
RETURN(E,F,v):
end:

> CellMatricize := proc(Cel,a,b,c)
  RETURN(Matrix(a,b*c,(i,j) ->
    Cel[i][floor((j-1)/c)+1,j-c*floor((j-1)/c)])):
end:

```

The procedure `CBNorm` is the main procedure; it calls upon the other procedures to compute the CB norm of the given map. The procedure begins by determining whether or not the given map is completely positive, and if so, returns the map's exact CB norm, using the result of Theorem 22. If the map is not completely positive, the algorithm described above begins to run.

Steps 1 through 3 are performed via the `MakeLinIndep` procedure. The minimization in step 4 is approximated by calling upon `RandomPositive` repeatedly to compute random positive matrices with eigenvalues in the interval $(0, 1]$ and taking the minimum of the resulting norm estimates. We can restrict ourselves to using only positive matrices with eigenvalues in the interval $(0, 1]$ rather than all positive matrices because multiplying such a matrix by a constant will not change the resulting norm estimate.

The inputs to the function are `CelA`, an array of the map's A operators in one of its generalized Choi-Kraus representations; `CelB`, an array of the map's B operators in the same representation; `NumIts`, the number of random matrices to be used to estimate the norm (a higher number will produce a more accurate estimate but will take longer to compute); and `NumOps`, the number of A and B operators that there are for the map in its given representation.

```

> CBNorm := proc(CelA,CelB,NumIts,NumOps)
  local i,x,n,k,CBGuess,NewCBGuess,ST,G,H,GG,HH,GGCB,HHCB,EF:
  n:=ColumnDimension(A[1]):k:=RowDimension(A[1]):
  CBGuess:=infinity:

  if IsCPMap(CelA,CelB,NumOps,n,k) then
    CBGuess:=Norm(add(MatrixMatrixMultiply(
      MatrixMatrixMultiply(A[x],IdentityMatrix(k)),B[x]),
      x=1..NumOps),2):
  else
    EF:=MakeLinIndep(CelA,CelB,NumOps,n,k):

    for i from 1 to NumIts do
      ST:=RandomPositive(EF[3],0,1):

      for x from 1 to EF[3] do
        H[x]:=Matrix(n,k,(i,j) ->
          add(ST[1][x,1]*EF[2][1][i,j],l=1..EF[3])):
        G[x]:=Matrix(k,n,(i,j) ->
          add(ST[2][1,x]*EF[1][1][i,j],l=1..EF[3])):
        HH[x]:=MatrixMatrixMultiply(
          HermitianTranspose(H[x]),H[x]):
        GG[x]:=MatrixMatrixMultiply(
          G[x],HermitianTranspose(G[x])):
      od:

      HHCB:=simplify(Matrix(k,k,(i,j) ->
        add(HH[1][i,j],l=1..EF[3]))):
      GGCB:=simplify(Matrix(k,k,(i,j) ->
        add(GG[1][i,j],l=1..EF[3]))):
      NewCBGuess:=Re(sqrt(evalf(Norm(GGCB,2)) *
        evalf(Norm(HHCB,2)))):
      if NewCBGuess < CBGuess then
        CBGuess:=NewCBGuess:
      fi:
    od:
  fi:

```

```

RETURN(CBGuess):
end:

```

To use the provided code, run the command `with(LinearAlgebra):` in Maple and then load in the procedures defined above. As an illustration we return to a special case of the class of maps discussed above.

Example 27. Let U be the 3×3 diagonal unitary matrix with eigenvalues $e^{(\frac{5i\pi}{4})}$, $e^{(i\pi)}$, and $e^{(\frac{3i\pi}{4})}$. Then the following code computes the CB norm of the map $\phi(X) = U^\dagger X U - X$.

```

> NumOps:=2:
> NumIts:=100:
> A:=Array(1..NumOps):
> B:=Array(1..NumOps):

> A[1]:=DiagonalMatrix([exp(-5*I*Pi/4),exp(-I*Pi),
exp(-3*I*Pi/4)]):
> A[2]:=IdentityMatrix(3):
> B[1]:=DiagonalMatrix([exp(5*I*Pi/4),exp(I*Pi),
exp(3*I*Pi/4)]):
> B[2]:=-IdentityMatrix(3):

> CBNorm(A,B,NumIts,NumOps);

```

Running this code gives an output of 1.449 in just under 7 seconds. Theorem 26 tells us however that $\|\phi\|_{cb} = \sqrt{2}$, so our algorithm is correct to two significant digits. To get a more accurate estimate, we can of course increase the number of iterations from 100, and it should be clear how to modify this code to find the CB norm of other maps.

5.1. Efficiency. To look at the efficiency of the algorithm, we assume that $n = k$ and consider Steps 1 - 3 separately from Step 4, as Steps 1 - 3 need only be run once for a given map, while we may wish to run our implementation of Step 4 hundreds or thousands of times for the same map.

First note that the most efficient algorithm that could possibly exist for computing the CB norm of a general CB map is $O(n^4)$, which can be seen by observing that a general CB map will have about n^2 linearly independent generalized Choi-Kraus operators, each of which has n^2 entries that must each be read at least once. One can observe, however, that the efficiency of Steps 1 - 3 of this algorithm is $O(n^8)$, as we need to apply Gaussian Elimination to about n^2 matrices each of dimension about $n^2 \times n^2$.

If we know already that the generalized Choi-Kraus operators of our given representation are linearly independent, we can simply proceed to Step 4 of the algorithm, which has efficiency that can be seen to be $O(n^5)$, as the step of computing $\sum_i G_i G_i^*$ and $\sum_i H_i^* H_i$ from the families of matrices $\{G_i\}$ and $\{H_i\}$ is $O(n^5)$ (if computed using the standard matrix multiplication

algorithm). Since all of the other operations in Step 4 are at least as efficient as $O(n^4)$, it follows that if we can find a clever way to compute $\sum_i G_i G_i^*$ and $\sum_i H_i^* H_i$, we can reduce the overall order of this step.

One of the most obvious ways to improve the order of Step 4 is to use an algorithm like the Strassen algorithm or the Coppersmith-Winograd algorithm [7] to perform our matrix multiplications. Employing these matrix multiplication techniques would then reduce the order of Step 4 to about $O(n^{4.807})$ or $O(n^{4.376})$, respectively. However, the resolution of either of two conjectures in [6] would imply that general matrix multiplication can be carried out in about $O(n^2)$ time, which would imply that Step 4 can be carried out in about $O(n^4)$ time, making our algorithm optimal for maps in which we already have a linearly independent representation.

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REFERENCES

- [1] D. Aharonov, A. Kitaev, N. Nisan, *Quantum circuits with mixed states*, Proc. 30th ACM Symposium on Theory of Computation (STOC) (1997), 20-30.
- [2] W.B. Arveson, *Subalgebras of C^* -algebras II*, Acta Math. **128** (1972), 271-308.
- [3] D.P. Blecher and V.I. Paulsen, *Tensor products of operator spaces*, J. Funct. Anal. **99** (1991), 262-292.
- [4] O. Bratteli, D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, 2nd edition (Springer, Berlin, 1987).
- [5] M. D. Choi, *Completely positive linear maps on complex matrices*, Lin. Alg. Appl. **10** (1975), 285-290.
- [6] Cohn, H., Umans, C., Kleinberg, R., Szegedy, B., *Group-theoretic Algorithms for Matrix Multiplication*, Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science 2005, IEEE Computer Society, 2005, 438-449.
- [7] D. Coppersmith and S. Winograd, *Matrix multiplication via arithmetic progressions*, J. Symbolic Computation, **9** (1990), 251-280.
- [8] K. R. Davidson, *C^* -algebras by example*, Fields Institute Monographs Amer. Math. Soc., Providence, 1996.
- [9] I. Devetak, M. Junge, C. King, M.B. Ruskai, *Multiplicativity of completely bounded p -norms implies a new additivity result*, arXiv:quant-ph/0506196 (2005).
- [10] F. Dupuis, P. Hayden, D. Leung, A. Winter, *The maximal p -norm multiplicativity conjecture is really, really false*, in preparation, 2007.
- [11] A. Gilchrist, N.K. Langford, M. Nielsen, *Distance measures to compare real and ideal quantum processes*, Phys. Rev. A **71**, 062310 (2005).
- [12] P. Hayden, *The maximal p -norm multiplicativity conjecture is false*, arXiv:0707.3291 (2007).

- [13] D.A. Herrero, *Approximation of Hilbert space operators, Volume 1 Second edition*, Longman Scientific & Technical **224**, Essex, 1989.
- [14] A. Jencova, *A relation between completely bounded norms and conjugate channels*, arXiv:quant-ph/0601071 (2006).
- [15] T.F. Jordan, A. Shaji, E.C.G. Sudarshan, *Dynamics of initially entangled open quantum systems*, Phys. Rev. A **70**, 052110 (2004).
- [16] A.Yu. Kitaev, *Quantum computations: algorithms and error correction*, Russian Math. Surveys **52** (1997), 1191-1249.
- [17] K. Kraus, *General state changes in quantum theory*, Ann. Physics **64** (1971), 311-335.
- [18] D. Kretschmann, D. W. Kribs, R. Spekkens, *Complementarity of correctable and private subsystems*, in preparation, 2007.
- [19] D. Kretschmann, D. Schlingemann, R. F. Werner, *The information-disturbance tradeoff and the continuity of Stinespring's representation*, arXiv:quant-ph/0605009 (2006).
- [20] M. A. Nielsen and I. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge, 2000.
- [21] V. I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advanced Mathematics **78**, Cambridge University Press, Cambridge, 2003.
- [22] V.I. Paulsen, *Representations of Function Algebras, Abstract Operator Spaces, and Banach Space Geometry*, J. Funct. Anal., **109** (1992), 113-129.
- [23] V.I. Paulsen, *The Maximal Operator Space of a Normed Space*," Proc. Edin. Math. Soc. **39**(1996), 309-323.
- [24] V.I. Paulsen and C.-Y. Suen, *Commutant representations of completely bounded maps*, J. Operator Thy., **13** (1985), 87-101.
- [25] D. Perez-Garcia, M.M. Wolf, C. Palazuelos, I. Villanueva, M. Junge, *Unbounded violation of tripartite Bell inequalities*, arXiv:quant-ph/0702189 (2007).
- [26] G. Pisier, *Non-commutative vector valued L_p -spaces and p -summing maps*, Soc. Math. France, Asterisque **247** (1998), 1-131.
- [27] A. Shabani, D.A. Lidar, *Linear quantum error correction*, arXiv:0708.1953 (2007).
- [28] R.R. Smith, *Completely bounded maps between C^* -algebras*, J. London Math. Soc. **27** (1983), 157-166.
- [29] J. Watrous, private communication.
- [30] J. Watrous, *Notes on super-operator norms induced by Schatten norms*, Quant. Inf. & Comp. **5** (2005), 58-68.
- [31] A. Winter, *The maximal output p -norm of quantum channels is not multiplicative for any $p > 2$* , arXiv:0707.0402 (2007).
- [32] G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz*, J. Funct. Anal., **40** (1981), 127-150.

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