MATH*1200 – LAB #3 SOLUTION

Solution to: www.nathanieljohnston.com/math1200/lab3.pdf

We must show that for any $\varepsilon > 0$, there exists a natural number N such that $\big||\sqrt{T+1}-\sqrt{T}|-0\big| < \varepsilon$ whenever $T > N$.

We start with what we want to prove. Try not to get scared by the ugly algebra – all the steps are basic operations that you should be comfortable with.

$$\begin{aligned} \big||\sqrt{T+1} - \sqrt{T}| - 0\big| &< \varepsilon \\ \iff \big|\sqrt{T+1} - \sqrt{T}\big| &< \varepsilon \\ \iff (\sqrt{T+1} - \sqrt{T})^2 &< \varepsilon^2 \\ \iff (T+1) - 2\sqrt{T(T+1)} + T &< \varepsilon^2 \\ \iff 2T + 1 - 2\sqrt{T(T+1)} &< \varepsilon^2 \\ \iff -2\sqrt{T(T+1)} &< \varepsilon^2 - 2T - 1 \\ \iff 2\sqrt{T(T+1)} &> 2T + 1 - \varepsilon^2 \end{aligned}$$

Now assume that T is large enough that $2T + 1 - \varepsilon^2$ is positive (i.e., $T > \frac{\varepsilon^2 - 1}{2}$). Then

$$\begin{aligned} 2\sqrt{T(T+1)} &> 2T + 1 - \varepsilon^2 \\ \iff (2\sqrt{T(T+1)})^2 &> (2T + 1 - \varepsilon^2)^2 \\ \iff 4T(T+1) &> 4T^2 + 1 + \varepsilon^4 + 4T - 4T\varepsilon^2 - 2\varepsilon^2 \\ \iff 4T^2 + 4T &> 4T^2 + 1 + \varepsilon^4 + 4T - 4T\varepsilon^2 - 2\varepsilon^2 \\ \iff 4T\varepsilon^2 &> 1 - 2\varepsilon^2 + \varepsilon^4 \\ \iff T &> \frac{1 - 2\varepsilon^2 + \varepsilon^4}{4\varepsilon^2} \end{aligned}$$

Thus, we should choose $N \geq \max\left\{\frac{\varepsilon^2-1}{2}, \frac{1-2\varepsilon^2+\varepsilon^4}{4\varepsilon^2}\right\}$.

Note: we squared both sides of some inequalities throughout the proof – this is OK because every time we squared both sides, we already knew that both sides were positive.

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